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Irrationality and Some Historical Remarks on Euler's Number e

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Abstract

One of the most important constants of mathematics is the Euler's number e. Many world mathematicians have studied the properties of number e and this number has an irreplaceable place not only in mathematics but also in other scientific disciplines. In the first part of the article we will focus on the introduction of the number e from a historical view and we will show some of its important features. The irrationality of the number e was first proven by Leonhard Euler in 1737. We present one of the possible proofs in the second part of this article.

Keywords: Leonhard Euler, number e, irrational, proof, historical remarks

Classification: 11A99, 01A99

1 Introduction

The number e is a very significant mathematical constant whose history is relatively "young". The first effort to simplify mathematical calculations was the introduction of numbers' logarithms, which in the 16-th century was dealt with independently by amateur mathematicians John Napier (1550 - 1617) and Joost Bürgi (1552 - 1632).

John Napier was the first in 1614 to have published a table of logarithms of the values of goniometric functions of sine, cosine, and tangent, which were favourably evaluated by professional mathematicians. In 1615, the English mathematician Henry Briggs (1561 - 1630) replaced Napier's logarithm by the decimal logarithm that helped to the mass use of logarithms and also presented a certain approximation of the decimal logarithm of a number later referred to as e.

The Dutch physicist, mathematician and astronomer Christiaan Huygens (1629-1695) made further progress in 1661, when he defined a "logarithmic" curve (in today's terminology, the exponential curve in the equation $y = k \cdot a^x$), with calculations coming to the computation of the constant created by the decimal logarithm of the "today's" number e to 17 decimal places.

The "today's" number e was first discovered and defined by using the limits by addressing the compound interest issue in 1683 by the Swiss mathematician Jacob Bernoulli (1654 - 1705), who first evaluated the limit of the sequence $\left\{\left(1+\frac{1}{n}\right)^n\right\}_{n=1}^{\infty}$.

In 1690, the "today's" number was e named and marked b by the German philosopher and mathematician, Gottfried Wilhelm Leibniz. Leonhard Euler (1707 - 1783) was the first

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mathematician to use the label around 1727 "e" for the number e and for the first time it was published in 1736 in his work *Mechanica*. Leonhard Euler examined and revealed different features of the number e [1] and came to an approximation of the number e to 18 decimal places $e = 2{,}718281828459045235$. Euler was the first to prove that the number e is irrational. In the next few years, many mathematicians tried to express the number e to as many decimal places as possible and to find its different properties. At present, the number e is expressed in millions of decimal places [2].

The number e is called

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2{,}71828182845904523536028747135 2 \ldots$$

The number e has some characteristic properties among all positive real numbers. First, we show that the number e is the only positive real number that meets the inequality (with the unknown a):

$$a^x \geq 1 + x$$

for each $x \in \mathbb{R}$. To show this relationship we will need a lemma.

Lemma. *Let be $d, x \in \mathbb{R}$, $d > -1$, $x \geq 1$. Then, it holds that*

$$(1+d)^x \geq 1 + xd.$$

The proof can be found in [3].

1. Let the number $a > 0$ have that property, that $a^x \geq 1 + x$ it applies to every $x \in \mathbb{R}$. We can show that $a = e$. Let be $x = \frac{1}{n}$ for $= 1,2,\cdots$. Then

$$a^{\frac{1}{n}} \geq 1 + \frac{1}{n}$$

From this

$$a \geq \left(1+\frac{1}{n}\right)^n = s_n, \qquad n = 1,2,\cdots$$

If we put $x = -\frac{1}{n+1}$, $n = 1,2,\cdots,t$hen

$$a^{-\frac{1}{n+1}} \geq 1 - \frac{1}{n+1} = \frac{1}{1+\frac{1}{n}}$$

$$\frac{1}{a^{\frac{1}{n+1}}} \geq \frac{1}{1+\frac{1}{n}} \Rightarrow 1 + \frac{1}{n} \geq a^{\frac{1}{n+1}}$$

From that

$$a \leq \left(1+\frac{1}{n}\right)^{n+1} = t_n, n = 1,2,\cdots$$

Then from the inequalities $a \geq \left(1+\frac{1}{n}\right)^n$, $a \leq \left(1+\frac{1}{n}\right)^{n+1}$ we have

$$\left(1+\frac{1}{n}\right)^n \leq a \leq \left(1+\frac{1}{n}\right)^{n+1}.$$

By $n \to \infty$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \leq \lim_{n\to\infty} a \leq \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}.$$

Since

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1},$$

based on the fundamental properties of the limits [4] we get $a = e$.

2. We show that the number e has the property that for each $x \in \mathbb{R}$ it holds

$$e \geq 1 + x$$

For $x = 0$, the relationship is valid. Let be $x > 0$ and let us determine a natural number n so that it holds $n \geq \frac{1}{x}$. Then $nx \geq 1$ and based on the relationship $(1+d)^x \geq 1 + xd$ for $d = \frac{1}{n}$ and the inequalities $\left(1+\frac{1}{n}\right)^{n} \leq e \leq \left(1+\frac{1}{n}\right)^{n+1}$ $n = 1,2,\cdots$, we obtain:

$$e^x \geq \left(1+\frac{1}{n}\right)^{nx} \geq 1 + nx\frac{1}{n} = 1 + x$$

Let be $x < 0$. Then $-\frac{1}{x} > 0$. Let us determine a natural number $n > 1$ so that it holds $n \geq -\frac{1}{x}$. Then we have $-nx \geq 1$ and, again on the basis of the relationship $(1+d)^x \geq 1 + xd$ for $=$ $-\frac{1}{n}$, we derive that

$$\left(1-\frac{1}{n}\right)^{-nx} \geq 1 + x.$$

For $n > 1$, it holds

$$\left(1-\frac{1}{n}\right)^{-n} = \left(\frac{n-1}{n}\right)^{-n} = \left(\frac{n}{n-1}\right)^{n} = \left(\frac{1+(n-1)}{n-1}\right)^{n} = \left(1+\frac{1}{n-1}\right)^{n} \geq e.$$

As $x < 0$, by powering the inequality $\left(1-\frac{1}{n}\right)^{-n} \geq e$ to $-x$, we receive

$$e^{-x} \leq \left(1-\frac{1}{n}\right)^{nx}.$$

Furthermore, by powering to the value -1 we get the following

$$e^x \geq \left(1-\frac{1}{n}\right)^{-nx} \geq 1 + x,$$

what completes the proof.

For the number e it holds

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n},\ x \in \mathbb{R}.$$

Let be $x \in \mathbb{R}$. Let be chosen $n_0 \in \mathbb{N}$ so that for each $n > n_0$ the inequality $\left|\frac{x}{n}\right| < 1$ holds.
In accordance with the preceding, the inequality $e^t \geq 1 + t$ holds for any $t \in \mathbb{R}$. Let's pose $t = \frac{x}{n}$, $n > n_0$. Then

$$e^{\frac{x}{n}} \geq 1 + \frac{x}{n} > 0.$$

By powering to n, we receive

$$e^x \geq \left(1 + \frac{x}{n}\right)^n.$$

Let be n a natural number greater than $-x$. Then, by substituting for $t = -\frac{x}{n+x}$ (for $n > n_0$ and $n > -x$) into the inequality $e^t \geq 1 + t$, we get

$$e^{-\frac{x}{n+x}} \geq 1 - \frac{x}{n+x}.$$

Next, let's edit the right side

$$1 - \frac{x}{n+x} = \frac{n+x}{n+x} - \frac{x}{n+x} = \frac{n}{n+x} = \frac{1}{1+\frac{x}{n}} > 0.$$

Hence

$$e^{-\frac{x}{n+x}} \geq \frac{1}{1+\frac{x}{n}}.$$

Let us power both sides of the inequality to n. Then

$$e^{-\frac{nx}{n+x}} \geq \frac{1}{\left(1 + \frac{x}{n}\right)^n}$$

and from that

$$\frac{1}{e^{\frac{nx}{n+x}}} \geq \frac{1}{\left(1+\frac{x}{n}\right)^n} \Rightarrow e^{\frac{nx}{n+x}} \leq \left(1 + \frac{x}{n}\right)^n \leq e^x.$$

By $n \to \infty$

$$\lim_{n\to\infty} e^{\frac{nx}{n+x}} \leq \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n \leq \lim_{n\to\infty} e^x.$$

If $n \to \infty$, then given the continuity of the function $g: \mathbb{R} \to \mathbb{R}\ g(t) = e^t$, we get

$$\lim_{n\to\infty} e^{\frac{nx}{n+x}} = e^x$$

and based on the fudamental characteristics of the limits [4], it holds

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x.$$

2 Number e is irrational

Finally, we will show that the number e is irrational [3].

Let us consider the sequence $\{a_n\}_{n=1}^{\infty}$, where $a_n = 1 + \frac{1}{1!} + \cdots + \frac{1}{n!}$, $n = 1,2,\cdots$. This sequence is an increasing one. Then, for each $n > 2$, it holds

$$n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n > \underbrace{2 \cdot 2 \cdot \cdots \cdot 2}_{(n-1)\ times} = 2^{n-1}.$$

Hence, $\frac{1}{n!} < \frac{1}{2^{n-1}}$. That's implies that $a_n < 2 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} < 3$ for any natural number $n > 2$. The sequence $\{a_n\}_{n=1}^{\infty}$ is therefore also bounded from above. Then $\lim_{n\to\infty} a_n$ exists and is equal to $\sup_{n\in\mathbb{N}} a_n$.

Now, we will indirectly show that the limit of this sequence is not a rational number.

Let suppose that $\lim_{n\to\infty} a_n = \frac{a}{b} \in \mathbb{Q}$ exists and should be $b > 3$ ($\frac{a}{b}$ does not need to be in a canonical form). For any $n \in \mathbb{N}$ holds that $\frac{a}{b} > 1 + \frac{1}{1!} + \cdots + \frac{1}{n!}$ and so for $\varepsilon > 0$ due to the definition of the sequence limit, it holds that for all n from a certain starting point that $\left|a_n - \frac{a}{b}\right| = \frac{a}{b} - a_n < \varepsilon$ holds.

Especially, the $\varepsilon = \frac{1}{2(b!)}$ holds for $0 < \frac{a}{b} - a_n < \frac{1}{2(b!)}$ big enough n (hence, we can already assume that $b < n$). Then

$$0 < \frac{a}{b} - \left(1 + \frac{1}{1!} + \cdots + \frac{1}{n!}\right) = \frac{a}{b} - \left(1 + \frac{1}{1!} + \cdots + \frac{1}{b!}\right) - \\ - \left(\frac{1}{(b+1)!} + \frac{1}{(b+2)!} + \cdots + \frac{1}{n!}\right) < \frac{1}{2(b!)}.$$

We multiply it by a number $b!$ and we get

$$b!\left[\frac{a}{b} - \left(1 + \frac{1}{1!} + \cdots + \frac{1}{b!}\right)\right] - \left[\frac{b!}{(b+1)!} + \frac{b!}{(b+2)!} + \cdots + \frac{b!}{n!}\right] < \frac{1}{2}.$$

Let's mark $c = b!\left[\frac{a}{b} - \left(1 + \frac{1}{1!} + \cdots + \frac{1}{b!}\right)\right] \geq 1$. For the second element it holds

$$\frac{b!}{(b+1)!} + \frac{b!}{(b+2)!} + \cdots + \frac{b!}{n!} = \\ = \frac{1}{b+1} + \frac{1}{(b+1)(b+2)} + \cdots + \frac{1}{(b+1)\cdots n} = \\ = \frac{1}{b+1}\left[1 + \frac{1}{b+2} + \cdots + \frac{1}{(b+2)\cdots n}\right] < \frac{1}{b+1}\left[1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-b-1}}\right] < \frac{2}{b+1}.$$

On the basis of this estimate, we are getting $1 \leq c < \frac{1}{2} + \frac{2}{b+1}$.

As $b > 3$, it holds that $1 \leq c < 1$, which is a contradiction and the number to which the sequence $\{a_n\}_{n=1}^{\infty}$ converges is not rational.

We can prove now that $\lim_{n\to\infty}\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right) = e.$

Let be $a_n = \left(1 + \frac{1}{n}\right)^n$, $b_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$, $n \in \mathbb{N}$. We will show that $\lim_{n\to\infty} b_n = e$.

We already know that $\left(1 + \frac{1}{n}\right)^n < + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}.$

We can show that $b_n \leq e$ for any $n \in \mathbb{N}$. Let k is any natural number, however, fixed in the next consideration. Then for any $n \in \mathbb{N}, k < n$, it holds

$$c_n = 2 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \cdots + \frac{1}{k!}\left(1 - \frac{1}{n}\right) \cdot \cdots \cdot \left(1 - \frac{k-1}{n}\right) < \\ < 2 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \cdots + \frac{1}{k!}\left(1 - \frac{1}{n}\right) \cdot \cdots \cdot \left(1 - \frac{k-1}{n}\right) + \cdots + \frac{1}{n!}\left(1 - \frac{1}{n}\right) \cdot \cdots \cdot \\ \cdot \cdots \cdot \left(1 - \frac{n-1}{n}\right) = \left(1 + \frac{1}{n}\right)^n < e.$$

We complete the definition of $c_1 = c_2 = \cdots = c_k = 1$. Then

$$\lim_{n\to\infty} c_n = 2 + \lim_{n\to\infty} \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \cdots + \lim_{n\to\infty} \frac{1}{k!}\left(1 - \frac{1}{n}\right) \cdot \cdots \cdot$$
$$\cdot \cdots \cdot \left(1 - \frac{k-1}{n}\right) = 2 + \frac{1}{2!} + \cdots + \frac{1}{k!} \leq e.$$

For each natural number n, it holds $a_n < b_n \leq e$. Then for $n \to \infty$ we get

$$e = \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n \leq e.$$

Hence, $\lim\limits_{n\to\infty}\left(1 + \frac{1}{1!} + \cdots + \frac{1}{n!}\right) = e$ and it menas that the number e is irrational.

Conclusion

In the paper we showed the proof that the number e is irrational and introduced some important historical remarks on number e. It can be also shown that the number e is not algebraic but transcendent [3][5].

The number e has a wide range of applications, whether in differential and integral calculus, in physical or economic applications. The number e also has many other interesting properties. For example it can also be expressed in the form of a non-terminating continued fraction [6]

$$e = [2, a_1, a_2, \cdots, a_k, \cdots],$$

where $a_{3m} = a_{3m-2} = 1$, $a_{3m-1} = 2m$, $m = 1,2,\cdots$. Hence

$$e = [2,1,2,1,1,4,1,1,6,1,1,8,1,1,10,1,1,12,\cdots] =$$
$$= 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cdots}}}}}$$

and then the irrationality of number e follows from the shape of its development into the continued fraction.

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