Acta Mathematica Nitriensia
Vol. 4, No. 1, p. 19 – 23
ISSN 2453-6083

Note to Ivory's Theorem

Dusan Vallo 1[a]

[a*] *Department of Mathematics, Faculty of Natural Sciences, Constantine the Philosopher University in Nitra, Tr. A. Hlinku 1, SK-949 74 Nitra,*

Received 24 April 2018; received in revised form 30 April 2018; accepted 1 May 2018

Abstract

In this paper we present a simple proof of the Ivory's Theorem for 2-dimensional case in Euclidean geometry.

Keywords: confocal conics, elliptical coordinates, Ivory's Theorem.

Classification: G10, R20

Introduction

Ivory's Theorem is one of the famous statements with wide applications in Physics and due to exist many generalizations, tabular referred in [1]. We concerned with proof of the simplest 2-dimensional version. To prove it, we use an elliptical coordinate system [2]. From this point of view, in this paper the problem is analyzed like a didactical note to conics and their attractive properties.

Confocal conics

Confocal conics are conics sharing the focuses. The set of confocal conics (ellipse, hyperbola) is states by equation

$$\frac{x^2}{a^2-\lambda}+\frac{y^2}{b^2-\lambda}=1, \tag{1}$$

where a is the semimajor axis, b is the semiminor axis and $\lambda \in R$ is parameter. We can suppose that $b^2 < a^2$. It holds true that

a) for $-\infty < \lambda < b^2$ the equation (1) represents an ellipse,

b) for $b^2 < \lambda < a^2$ the equation (1) represents a hyperbola.

Short history of the Ivory's Theorem

The original version of this theorem deals with confocal quadrics. British mathematician James Ivory (1765-1842) computed the gravitational field created by a solid homogenous ellipsoid. In particular, French mathematician Michel F. Chasles (1793-1880) formulated the statement what is now famous like the Ivory's Theorem. [3]

Theorem

The diagonals of a quadrilateral made by arcs of confocal ellipses and hyperbolas are equal.

---

*Corresponding author: dvallo@ukf.sk
DOI: 10.17846/AMN.2018.4.1.19-23

Proof. Let us consider two confocal ellipses

$$k_1 \coloneqq \frac{x^2}{a^2-\lambda_1}+\frac{y^2}{b^2-\lambda_1}=1, \quad k_3 \coloneqq \frac{x^2}{a^2-\lambda_3}+\frac{y^2}{b^2-\lambda_3}=1, \tag{2}$$

for $0<b<a, -\infty<\lambda_1<b^2, -\infty<\lambda_3<b^2,\ \lambda_1\neq\lambda_3$.

The confocal hyperbolas corresponding to the focuses of the ellipses (2) have equations

$$k_2 \coloneqq \frac{x^2}{a^2-\lambda_2}+\frac{y^2}{b^2-\lambda_2}=1, \quad k_4 \coloneqq \frac{x^2}{a^2-\lambda_4}+\frac{y^2}{b^2-\lambda_4}=1, \tag{3}$$

for $b^2<\lambda_2<a^2, b^2<\lambda_4<a^2,\ \lambda_2\neq\lambda_4$.

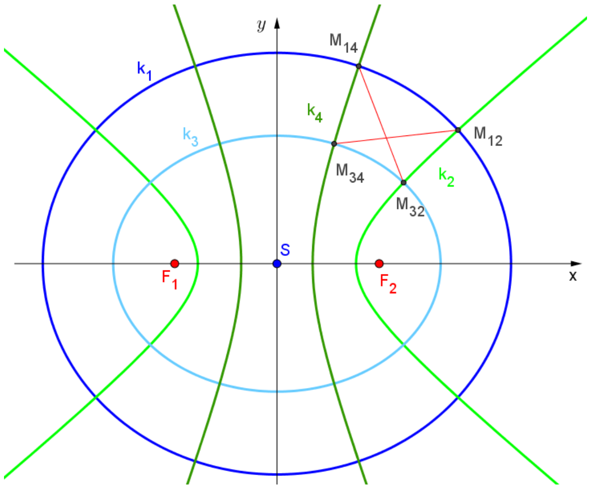

Fig. 1a

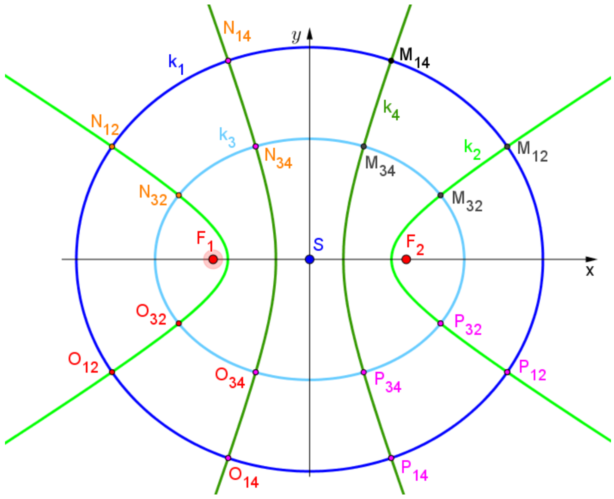

Fig. 1b

Let us fix the intersection points

$M_{12}=k_1\cap k_2, M_{14}=k_1\cap k_4,\ M_{32}=k_2\cap k_3,\ M_{34}=k_3\cap k_4$ in the 1st quadrant (see *Fig. 1a*). Their coordinates are in the form

$$M_{ij}[x_{ij},y_{ij}] = M_{ij}\left[\sqrt{\frac{(a^2-\lambda_i)(a^2-\lambda_j)}{a^2-b^2}},\sqrt{\frac{(b^2-\lambda_i)(b^2-\lambda_j)}{b^2-a^2}}\right]. \tag{4}$$

The labels of the coordinates of the points M_{ij} in (4) will be **fixed** for the purpose of the following computing.

1) At first we prove that $|M_{12}M_{34}|^2=|M_{14}M_{32}|^2$. Let us calculate

$$|M_{12}M_{34}|^2=(x_{12}-x_{34})^2+(y_{12}-y_{34})^2$$

$$|M_{12}M_{34}|^2=x_{12}^2+x_{34}^2+y_{12}^2+y_{34}^2-2(x_{12}x_{34}+y_{12}y_{34})$$

$$\vdots$$

$$|M_{12}M_{34}|^2=2(a^2+b^2)-\sum_{i=1}^{4}\lambda_i-2\left(\frac{\sqrt{\prod_{i=1}^{4}(a^2-\lambda_i)}}{|a^2-b^2|}+\frac{\sqrt{\prod_{i=1}^{4}(b^2-\lambda_i)}}{|b^2-a^2|}\right)$$

And finally we obtain

$$|M_{12}M_{34}|^2 = 2(a^2+b^2) - \sum_{i=1}^{4}\lambda_i - \frac{2}{a^2-b^2}\left(\sqrt{\prod_{i=1}^{4}(a^2-\lambda_i)} + \sqrt{\prod_{i=1}^{4}(b^2-\lambda_i)}\right) \quad (5)$$

The result is symmetrical in permutation of indices. It implies that $|M_{12}M_{34}|^2 = |M_{14}M_{32}|^2$.

2) The restriction for the curved quadrilateral $M_{12}M_{14}M_{34}M_{32}$ is not necessary.

Another curved quadrilaterals can be considered, e.g. $M_{12}N_{14}N_{34}M_{32}$, $M_{12}O_{14}O_{34}M_{32}$ and $M_{12}P_{14}P_{34}M_{32}$ (see *Fig. 2a, b, c*). We compute the square of the lengths in these cases.

Remark. Visually, in Fig. 2 we fix the curved side $M_{12}M_{32}$ and the point M_{14} resp. M_{34} "move along" the ellipse k_1, resp. ellipse k_2 to the corresponding intersections points.

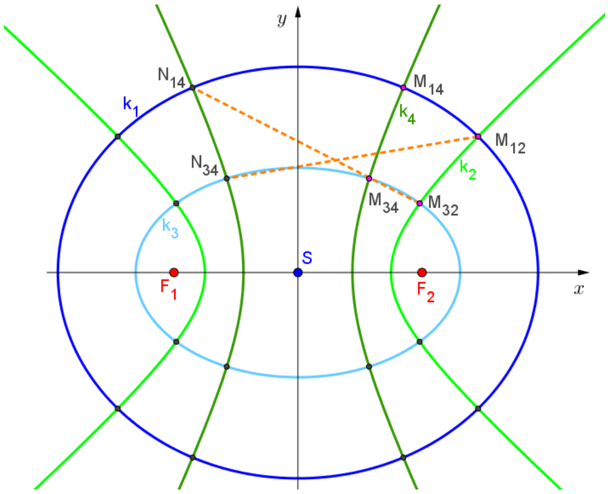

Fig. 2a

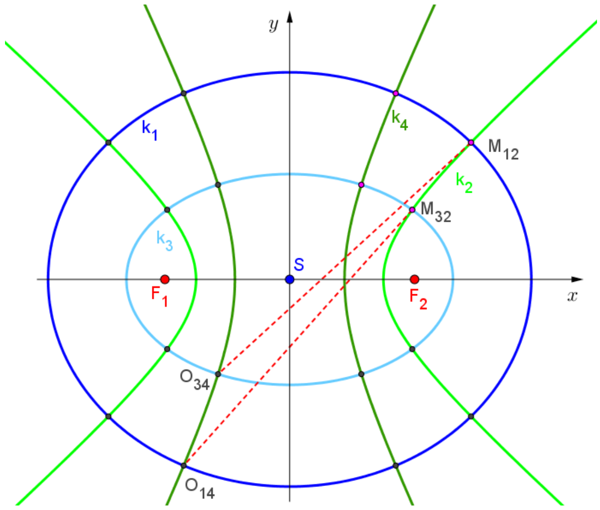

Fig. 2b

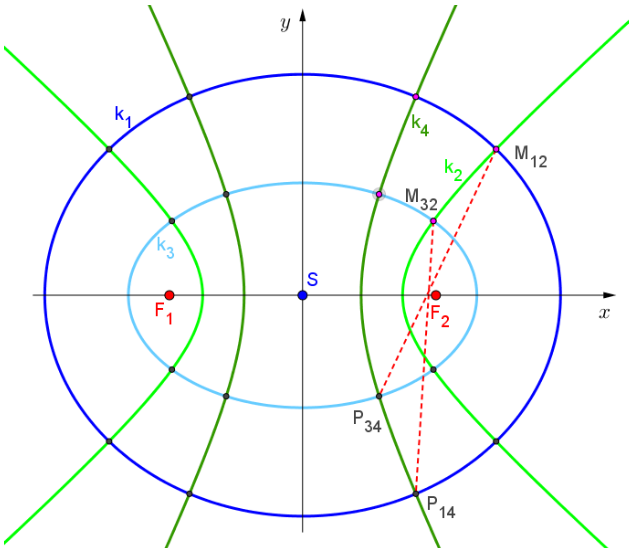

Fig. 2c

In previous notation and in usage of axial symmetry of conics, we derive that $N_{14}[-x_{14}, y_{14}]$, $N_{34}[-x_{34}, y_{34}]$. Holds

$$|M_{12}N_{34}|^2 = (x_{12} + x_{34})^2 + (y_{12} - y_{34})^2$$
$$|M_{12}N_{34}|^2 = x_{12}^2 + x_{34}^2 + y_{12}^2 + y_{34}^2 + 2(x_{12}x_{34} - y_{12}y_{34})$$
$$\vdots$$
$$|M_{12}N_{34}|^2 = 2(a^2 + b^2) - \sum_{i=1}^{4} \lambda_i + \frac{2}{a^2 - b^2}\left(\sqrt{\prod_{i=1}^{4}(a^2 - \lambda_i)} - \sqrt{\prod_{i=1}^{4}(b^2 - \lambda_i)}\right) \qquad (6)$$

By analogy, the result is symmetrical in permutation of indices. It implies that
$$|M_{12}N_{34}|^2 = |N_{14}M_{32}|^2.$$

Following the ideas, we obtain $O_{14}[-x_{14}, -y_{14}]$, $O_{34}[-x_{34}, -y_{34}]$ and holds
$$|M_{12}O_{34}|^2 = (x_{12} + x_{34})^2 + (y_{12} + y_{34})^2$$
$$\vdots$$
$$|M_{12}N_{34}|^2 = 2(a^2 + b^2) - \sum_{i=1}^{4} \lambda_i + \frac{2}{a^2 - b^2}\left(\sqrt{\prod_{i=1}^{4}(a^2 - \lambda_i)} + \sqrt{\prod_{i=1}^{4}(b^2 - \lambda_i)}\right) \qquad (7)$$

It also implies that
$$|M_{12}O_{34}|^2 = |O_{14}M_{32}|^2.$$

Finally, for the points $P_{14}[x_{14}, -y_{14}]$, $P_{34}[x_{34}, -y_{34}]$ we derive
$$|M_{12}P_{34}|^2 = (x_{12} - x_{34})^2 + (y_{12} + y_{34})^2$$
$$\vdots$$
$$|M_{12}P_{34}|^2 = 2(a^2 + b^2) - \sum_{i=1}^{4} \lambda_i + \frac{2}{a^2 - b^2}\left(-\sqrt{\prod_{i=1}^{4}(a^2 - \lambda_i)} + \sqrt{\prod_{i=1}^{4}(b^2 - \lambda_i)}\right) \qquad (8)$$

We obtain that
$$|M_{12}P_{34}|^2 = |P_{14}M_{32}|^2.$$

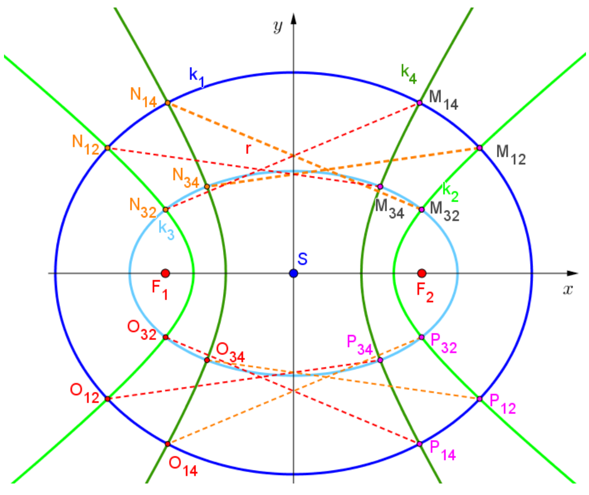

Fig. 3a

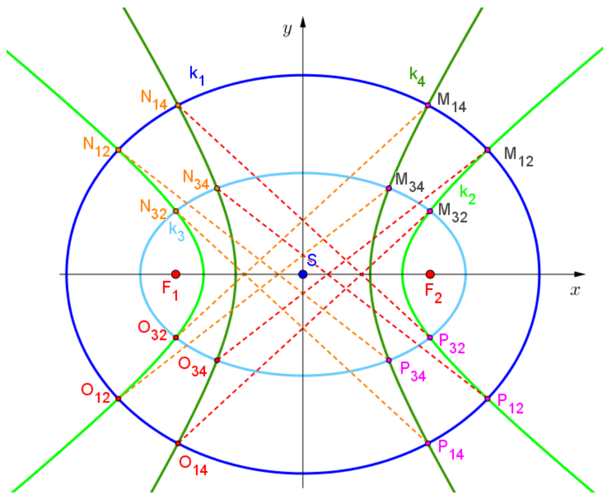

Fig. 3b

3) If we apply the axial symmetry of the conics, then we construct another curved

quadrilaterals which corresponding diagonals have equal lengths. In *Fig. 3a-c,* the situations related to fundamental positions of diagonals in *Fig. 2a-c* are drawn. The reader can verify that in all cases we obtain the one of the formulas in form (5) – (8). ■

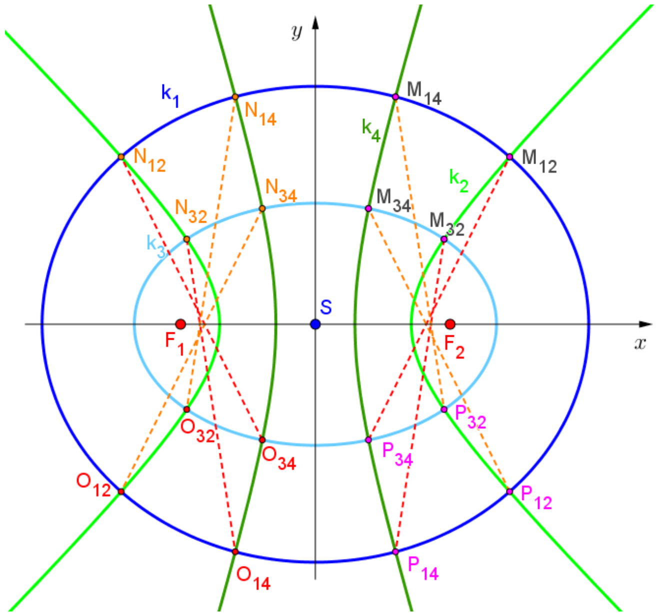

Fig. 3c

Conclusion

We proved the planar version of the famous Ivory's Theorem in Euclidean plane by using the confocal elliptical coordinates. The classical result for curved quadrilateral lying in the 1st quadrant was extended. We derived the formulas for square of the lengths of the diagonals of the fundamental quadrilaterals, too. Application of axial symmetries of the conics allows us to demonstrate that there exist only four different values of the lengths of the diagonals.

References

[1] Izmestiev Ivan, Tabachnikov Serge. Ivory's theorem revisited, *Journal of Integrable Systems*, Volume 2, Issue 1, 1 January 2017, xyx006. Available at: https://academic.oup.com/integrablesystems/article/2/1/xyx006/3939821

[2] Pírko Zdeňek; *O souřadnicích v rovine*. JČMF v Praze, Cesta k vědení, 15. sv., 1942

[3] Izmestiev Ivan. Spherical and hyperbolic conics. Available at: https://arxiv.org/abs/1702.06860.

[4] Weisstein, Eric W. "Confocal Ellipsoidal Coordinates." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/ConfocalEllipsoidalCoordinates.html