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Arithmetic Mean and Geometric Mean

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Abstract

In mathematics we define several types of means. Probably the most known are the arithmetic and geometric means. If we are given n nonnegative numbers $x_1, x_2, \ldots x_n$; then the number $A_n = \frac{x_1 + x_2 + \ldots + x_n}{n}$ we call the arithmetic mean and the number $G_n = \sqrt[n]{x_1 x_2 \ldots x_n}$ we call the geometric mean of the numbers given. In the first part of the paper with the use of functions´ their of more variables we will show that for each natural number n $A_n \geq G_n$ applies. In the second part we will try to show the same, however, without using the differential calculus.

Keywords: arithmetic mean, geometric mean, proof, differential calculus

Classification: E55

Introduction

When given n nonnegative numbers $x_1, x_2, \ldots x_n$. Then the number $A_n = \frac{x_1 + x_2 + \ldots + x_n}{n}$ we call the arithmetic mean of the numbers $x_1, x_2, \ldots x_n$. Number $G_n = \sqrt[n]{x_1 x_2 \ldots x_n}$ we call the geometric mean of numbers $x_1, x_2, \ldots x_n$. We will try to find the relation between numbers A_n a G_n.

Problem A1.

Divide the positive number k into two addends so their product would be the highest the possible.

Solution.

(See in [2]) Lets divide number k into two parts, which will be labeled as x a y $(0 \leq x \leq k;\ 0 \leq y \leq k)$. Product of this two addends should be as high as possible, that is we search the global maximum of the function $s(x, y) = xy$; while $x + y = k$ applies. We can see, extreme of the two variables function should be found $s = s(x, y)$. However, it will be easier to calculate the global maximum of the one variable function. $s = s(x) = x(k - x)$; where $x \in \langle 0; k \rangle$. When solving the equation $s'(x) = 0$ we will get the stationary point $u_0 = \frac{k}{2}$. With the help of the second

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derivation $s''(x)$ and comparison of the value $s\left(\frac{k}{2}\right)$ with values $s(0)$, $s(k)$ we will find out, that the function $s(x)$ acquires its highest value in point $u_0 = \frac{k}{2}$. It means, that the function $s(x, y)$ acquires global maximum in the point $X_0\left[\frac{k}{2}; \frac{k}{2}\right]$. So if $0 \le x$, $0 \le y$, $x + y = k$, then $xy \le \frac{k}{2} \cdot \frac{k}{2} = \left(\frac{k}{2}\right)^2 = \left(\frac{x+y}{2}\right)^2$. Thence implies, that

$$\sqrt{xy} \le \frac{x+y}{2}, \text{ this is } G_2 \le A_2.$$

Problem A2.

Divide the positive number k into three addends so their product would be the highest the possible.

Solution.

(See in [1]) Lets divide number k into three parts, which will be labeled as x, y a z ($0 \le x \le k$; $0 \le y \le k$; $0 \le z \le k$). Product of this three addends should be as high as possible, that is we search the global maximum of the function $s(x, y, z) = xyz$; while $x + y + z = k$ applies. Again, we will make the problem easier – we will search global maximum of the two variable function

$$s(x, y) = xy(k - x - y)$$

in the domain restricted by the lines $x = 0$, $y = 0$, $x + y = k$. When solving the scheme of equation

$$s'_x(x, y) = 0$$
$$s'_y(x, y) = 0$$

we will get stationary points $U_0\left[\frac{k}{3}; \frac{k}{3}\right]$; $U_1[0; k]$; $U_2[k; 0]$; $U_3[0; 0]$. It will be easily possible to show that the function $s(x, y)$ acquires its highest value at the point $U_0\left[\frac{k}{3}; \frac{k}{3}\right]$. It means that the function $s(x, y, z)$ acquires global maximum at the point $X_0\left[\frac{k}{3}; \frac{k}{3}; \frac{k}{3}\right]$. Therefore if $0 \le x$, $0 \le y$, $0 \le z$, then $xyz \le \frac{k}{3} \cdot \frac{k}{3} \cdot \frac{k}{3} = \left(\frac{k}{3}\right)^3 = \left(\frac{x+y+z}{3}\right)^3$ applies. Thence it follows that

$$\sqrt[3]{xyz} \le \frac{x+y+z}{3}, \text{ that is } G_3 \le A_3.$$

<u>Problem A3.</u>

Find the highest value of the n – extraction of the n positive numbers x_1, x_2, ... x_n product when conditioned that the sum of these numbers is equal to number k.

Solution.

(See in [1]) In the previous sum we have found the maximum of the function $s(x_1, x_2, \ldots x_n) = \sqrt[n]{x_1 \cdot x_2 \cdot \ldots \cdot x_n}$, while $x_1 + x_2 + \ldots + x_n = k$ applies. We say that by the equality $x_1 + x_2 + \ldots + x_n = k$ the bond is given, that is we calculate the fixed extreme of the function $s(x_1, x_2, \ldots x_n)$. We will set the Lagrange function

$$L(x_1, x_2, \ldots x_n, \lambda) = \sqrt[n]{x_1 \cdot x_2 \cdot \ldots \cdot x_n} + \lambda(x_1 + x_2 + \ldots + x_n - k).$$

Stationary points will be found when solving the system of equations

$$\begin{aligned}
&\frac{\partial L(x_1, x_2, \ldots x_n, \lambda)}{\partial x_1} = 0\\
&\frac{\partial L(x_1, x_2, \ldots x_n, \lambda)}{\partial x_2} = 0\\
&\vdots\\
&\frac{\partial L(x_1, x_2, \ldots x_n, \lambda)}{\partial x_n} = 0\\
&\frac{\partial L(x_1, x_2, \ldots x_n, \lambda)}{\partial \lambda} = 0
\end{aligned}$$

we obtain

$$\begin{aligned}
&\frac{\partial L}{\partial x_1} = \frac{1}{n}\frac{x_2 \cdot x_3 \cdot \ldots \cdot x_n}{(x_1 \cdot x_2 \cdot \ldots \cdot x_n)^{\frac{n-1}{n}}} + \lambda = \frac{1}{n}\frac{L}{x_1} + \lambda = 0 \quad \Rightarrow \quad L = -n\lambda x_1\\
&\frac{\partial L}{\partial x_2} = \frac{1}{n}\frac{x_1 \cdot x_3 \cdot \ldots \cdot x_n}{(x_1 \cdot x_2 \cdot \ldots \cdot x_n)^{\frac{n-1}{n}}} + \lambda = \frac{1}{n}\frac{L}{x_2} + \lambda = 0 \quad \Rightarrow \quad L = -n\lambda x_2\\
&\vdots\\
&\frac{\partial L}{\partial x_n} = \frac{1}{n}\frac{x_1 \cdot x_2 \cdot \ldots \cdot x_{n-1}}{(x_1 \cdot x_2 \cdot \ldots \cdot x_n)^{\frac{n-1}{n}}} + \lambda = \frac{1}{n}\frac{L}{x_n} + \lambda = 0 \quad \Rightarrow \quad L = -n\lambda x_n\\
&\frac{\partial L}{\partial \lambda} = x_1 + x_2 + \ldots + x_n - k = 0
\end{aligned}$$

Since $n\lambda x_1 = n\lambda x_2 = \ldots = n\lambda x_n$, the equality $x_1 = x_2 = \ldots = x_n$ have to apply. Thence out of the equation $\frac{\partial L}{\partial \lambda} = 0$ it follows that $x_1 = x_2 = \ldots = x_n = \frac{k}{n}$. It could be verified, that the function $s(x_1, x_2, \ldots x_n)$ acquires at the point $X_0\left[\frac{k}{n}; \frac{k}{n}; \ldots; \frac{k}{n}\right]$ fixed global maximum. That means

$$\sqrt[n]{x_1 \cdot x_2 \cdot \ldots \cdot x_n} \leq \sqrt[n]{\frac{k}{n} \cdot \frac{k}{n} \cdot \ldots \cdot \frac{k}{n}} = \frac{k}{n} = \frac{x_1 + x_2 + \ldots + x_n}{n}$$

applies, or $G_n \leq A_n$.

Problem B1.

Prove that for $\forall x_1, x_2 \in R: x_1 > 1 \wedge x_2 < 1 \Rightarrow x_1 + x_2 > x_1 x_2 + 1$.

Solution.

(See in [3]) Probably $x_2 < 1 \Leftrightarrow 1 - x_2 > 0$. Then $0 < 1 - x_2 < x_1(1 - x_2) \Leftrightarrow x_1 + x_2 > 1 + x_1 x_2$ due to $0 < 1 < x_1$, what had to be shown.

Problem B2.

Prove that for $x_i \in R^+$, $i = 1, 2, \ldots, n$, and $\prod_{i=1}^{n} x_i = 1$ applies $\sum_{i=1}^{n} x_i \geq n$.

Solution.

(See in [4]) The sum will be proved by mathematical induction:

a) n = 1: $x_1 = 1 \Rightarrow x_1 \geq 1$ trivially applies;

b) n = 2: $x_1 x_2 = 1 \Rightarrow x_1 + x_2 \geq 2 = 1 + x_1 x_2$ applies according to the sum B1, consider:

If $x_1 x_2 = 1$ then either $x_1 = x_2 = 1$ and the sum is trivial again and $x_1 + x_2 = 2$, or $x_1 \neq x_2$. Then $x_1 = 1/x_2 \Leftrightarrow x_2 = 1/x_1$ and without any loss on generality it is possible to suppose that e.g. $x_1 > 1 \wedge (0 <) x_2 < 1$. Then according to B1 it is obvious that $x_1 + x_2 > x_1 x_2 + 1$, however when $x_1 x_2 = 1$, then $x_1 + x_2 > 2$. When combining the trivial and the general parts we get the affirmation in the form for n = 2: $x_1 x_2 = 1 \Rightarrow x_1 + x_2 \geq 2 = 1 + x_1 x_2$.

c) Lets follow in the induction generally. When $x_i = 1$ for $\forall\ i = 1,\ 2.\ldots\ n$, the affirmation applies trivially (with the character of equality).

Let for $x_i \in R^+$, $i = 1, 2, \ldots, n$, holds $\prod_{i=1}^{k} x_i = 1 \Rightarrow \sum_{i=1}^{k} x_i \geq k$. Assume now that for $y_j \in R^+$, $j = 1, 2, \ldots, n + 1$, holds $\sum_{j=1}^{n+1} y_j = 1$. Without any loss of generality of the sum we will re-label the numbers, so $y_1 \leq 1$ and $y_2 \geq 1$.

Then $(1 - y_1)(1 - y_2) \leq 0$, or $1 + y_1 y_2 \leq y_1 + y_2$, and we obtain $1 + y_1 y_2 + y_3 + \cdots + y_{n+1} \leq y_1 + y_2 + y_3 + \cdots + y_{n+1}$.

Mark $x_1 = y_1 y_2,\ x_2 = y_3, \dots, x_n = y_{n+1}$. Then $\prod_{i=1}^{n} x_i = \prod_{j=1}^{n+1} y_j = 1$. On base of induction hypothesis $\sum_{i=1}^{n} x_i \geq n$. Then we have

$$\sum_{i=1}^{n} x_i = y_1 y_2 + \sum_{j=3}^{n+1} y_j \geq n \text{ and so } 1 + y_1 y_2 + \sum_{j=3}^{n+1} y_j \geq 1 + n.$$

But $y_1 + y_2 \geq 1 + y_1 y_2$, so

$$y_1 + y_2 + \sum_{j=3}^{n+1} y_j \geq 1 + n, \text{ finally } \sum_{j=1}^{n+1} y_j \geq 1 + n.$$

Problem B3.

Prove that for: $\forall\, x_i \in R^+$ pre $i = 1, 2, \dots n$ platí: $G_n = \sqrt[n]{\prod_{i=1}^{n} x_i} \leq \frac{1}{n}\sum_{i=1}^{n} x_i = A_n$ (thus the geometric mean G_n of positive real numbers x_i is smaller or at a most equal to the arithmetic mean A_n of these numbers).

Solution.

(See in [3]) If the inequality that have to be proved will be simply turned into the form $\frac{1}{n}\sum_{i=1}^{n} x_i \geq \sqrt[n]{\prod_{i=1}^{n} x_i}$ and multiplied by the term $\frac{n}{\sqrt[n]{\prod_{i=1}^{n} x_i}}$, we will get $\frac{\sum_{i=1}^{n} x_i}{\sqrt[n]{\prod_{i=1}^{n} x_i}} \geq n$. Lets label $\frac{x_i}{\sqrt[n]{\prod_{i=1}^{n} x_i}} = y_i$. Probably $\frac{\sum_{i=1}^{n} x_i}{\sqrt[n]{\prod_{i=1}^{n} x_i}} = \sum_{i=1}^{n} y_i$, where $y_i > 0$ for $\forall\, i = 1, 2 \dots n$. Finally, the reader will consider that $\prod_{i=1}^{n} y_i = \prod_{i=1}^{n}\left(\frac{x_i}{\sqrt[n]{\prod_{i=1}^{n} x_i}}\right) = \frac{\prod_{i=1}^{n} x_i}{\left(\sqrt[n]{\prod_{i=1}^{n} x_i}\right)^n} = \frac{\prod_{i=1}^{n} x_i}{\prod_{i=1}^{n} x_i} = 1$, so the numbers $y_i = \frac{x_i}{\sqrt[n]{\prod_{i=1}^{n} x_i}}$ meet the requests from the sum B2 thanks to what the proved inequality $G_n \leq A_n$ applies.

Conclusion

Using two different strategies (based on applying of differential calculus of function of more variables and using mathematical induction too) we have shown a well known inequality between geometric mean and arithmetic mean. We are sure there could be illustrated further extension and generalization of this relation. Namely more complex inequalities could be demonstrated using college knowledge, such as $H_n \leq G_n \leq A_n \leq K_n$, where $H_n = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}$ is harmonic mean and $K_n = \sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2}$ is quadratic mean.

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