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Limit of a Sequence Defined by Recursion

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Abstract

Limit of a sequence is one of the fundamental notions in university mathematics. Computing limits of various types is included in standard tasks. This paper is devoted to selected non-standard and/or problem limits. The sequences in question are defined by recursion, thus traditional algorithms are of no use here and must be superseded by other methods.

Keywords: sequence defined by recursion, limit of sequence, application of limit of sequence

Classification: I35

Introduction

Limit of a sequence is one of the cornerstones of calculus. Based on this notion calculus defines the sum of infinite series, limit of a function (Heine definition), then derivative of a function (special type of a limit of a function) and finally also definite (Riemann) integral.

Consequently, the main mathematical skills of university students who attend calculus courses must include also computation of various types of limits of a sequence. The pillars are the limits which may be referred to as paradigm limits, such as

$$\lim_{n\to\infty} n^k = \infty, \quad \lim_{n\to\infty} \frac{1}{n^k} = \infty \quad \text{(both for } k>0\text{)}, \quad \lim_{n\to\infty} \sqrt[n]{a} = 1 \quad \text{(where } a>0\text{)}, \quad \lim_{n\to\infty} \sqrt[n]{n} = 1,$$

$$\lim_{n\to\infty} \frac{a^n}{n^k} = \infty \quad \text{(for } a>1,\ k>0\text{)}, \qquad \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e, \text{ and } \qquad \lim_{n\to\infty} q^n = \begin{cases} \infty; & q>1 \\ 1; & q=1 \\ 0; & |q|<1 \end{cases}.$$

As far as the computation is concerned, the simplest types of limits include the limit of a quotient of two polynomials. The method appropriate for its computation – dividing both the numerator and the denominator by the expression which “approaches infinity the fastest” – can also be generalized for computation of other limits of ∞/∞ type. When dealing with limits of $\infty-\infty$ type, which are often encountered in form of a difference of two roots, it is usually advantageous to extend the expression by means of multiplication by an appropriate fraction. Last but not least, there is a plentiful group of limits of 1^∞ type, and those can be solved by changing them into the form of number e, i. e. Euler‘s number.

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All of the above mentioned methods should be mastered by an ordinary student, as they belong to standard tasks requiring only appropriate computational algorithms. So far we have not mentioned one important property of the previous examples – the sequences are defined in analytic way.

A more challenging task is to find the limit of a sequence defined by recursion. Several examples are presented and discussed further in this paper.

Existence and computation of limits

For the purposes of computing the limits of sequences defined by recursion there is no general template-like algorithm. Yet, the main idea might be summarized in two steps – find out whether the limit of the given sequence exists, and then, if it exists, compute the limit. Firstly, let us note two important theorems which we refer to in the below presented examples.

Weierstrass theorem (*Wt1*). Let sequence $\{a_n\}_{n=1}^{\infty}$ be non-decreasing. If the sequence:

1) is not bounded above, then $\lim\limits_{n\to\infty} a_n = \infty$;

2) else is bounded above, then $\lim\limits_{n\to\infty} a_n = \sup\{a_n;\ n\in\mathbb{N}\}$.

Weierstrass theorem (*Wt2*). Let sequence $\{a_n\}_{n=1}^{\infty}$ be non-increasing. If the sequence:

1) is not bounded below, then $\lim\limits_{n\to\infty} a_n = -\infty$;

2) else is bounded below, then $\lim\limits_{n\to\infty} a_n = \inf\{a_n;\ n\in\mathbb{N}\}$.

Cauchy – Bolzano convergence criterion (*CBcc*). Sequence $\{a_n\}_{n=1}^{\infty}$ is convergent if and only if for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n, m > n_0$, $n, m \in \mathbb{N}$, it holds that

$$|a_n - a_m| < \varepsilon.$$

Example 1.

Find out whether the limit of sequence $\{x_n\}_{n=1}^{\infty}$ exists, where $x_1 = \sqrt{k}$, $x_{n+1} = \sqrt{k + x_n}$; where $k > 0$ is a given constant.

Solution. Our hypothesis implies that the given sequence is increasing – hereby we prove it by induction.

Let $n = 1$. Obviously $k < k + \sqrt{k}$; then also $x_1 = \sqrt{k} < \sqrt{k+\sqrt{k}} = x_2$. Assume now that $x_{n-1} < x_n$. Then $k + x_{n-1} < k + x_n$, and it also holds that $x_n = \sqrt{k + x_{n-1}} < \sqrt{k + x_n} = x_{n+1}$.

Furthermore, we claim that the sequence in question is bounded above. The relation $x_n = \sqrt{k + x_{n-1}}$ suggests that $x_n^2 = k + x_{n-1}$, also $x_n = \frac{k}{x_n} + \frac{x_{n-1}}{x_n}$. The first part implies that $x_1 < x_n$, therefore $\frac{k}{x_n} < \frac{k}{x_1}$. Next, it also holds that $x_{n-1} < x_n$, i. e. $\frac{x_{n-1}}{x_n} < 1$. Thus, we obtain the upper bound: $x_n = \frac{k}{x_n} + \frac{x_{n-1}}{x_n} < \frac{k}{x_1} + 1 = \frac{k}{\sqrt{k}} + 1 = \sqrt{k} + 1$.

Now, let us add another inductive proof. For $n = 1$ it obviously holds that $x_1 = \sqrt{k} < \sqrt{k} + 1$. Next, assume that $x_n < \sqrt{k} + 1$; then

$$x_{n+1} = \sqrt{k + x_n} < \sqrt{k + \left(\sqrt{k} + 1\right)} < \sqrt{k + 2\sqrt{k} + 1} = \sqrt{\left(\sqrt{k} + 1\right)^2} = \sqrt{k} + 1.$$

Since sequence $\{x_n\}_{n=1}^{\infty}$ is increasing and bounded above, according to theorem (*Wt1*) it is convergent. Mark $\alpha = \lim_{n\to\infty} x_n$. Having applied the limit transition we obtained $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \sqrt{k + x_{n-1}}$, i. e. $\alpha = \sqrt{k + \alpha}$. Thus, we have quadratic equation $\alpha^2 - \alpha - k = 0$, and the conditions of the task (the terms of the sequence are non-negative) are met only by solution $$\alpha = \frac{1}{2}\left(1 + \sqrt{1 + 4k}\right).$$

Example 2.

Show that sequence $\left\{\frac{F_{n+1}}{F_n}\right\}_{n=1}^{\infty}$, where F_n are Fibonacci numbers, is convergent.

Solution. The terms of Fibonacci sequence are defined by recursion as follows

$$F_1 = 1,\ F_2 = 1,\ F_n = F_{n-2} + F_{n-1}.$$

Let $\varepsilon > 0$. According to Archimedean property of natural numbers there exists $p \in \mathbb{N}$ such that $\frac{1}{p} < \varepsilon$. However, let us assume $p > 4$ in order to ensure that $F_n > n$ for $n > p$.

If $m = n$, then $\left|\frac{F_{n+1}}{F_n} - \frac{F_{m+1}}{F_m}\right| = 0 < \varepsilon$, and the proof is done. Thus, let be $m \neq n$, maintaining the generality, it may be assumed that $m > n$.

Compute:

$$\left|\frac{F_{n+1}}{F_n}-\frac{F_{m+1}}{F_m}\right|=\left|\left(\frac{F_{n+1}}{F_n}-\frac{F_{n+2}}{F_{n+1}}\right)+\left(\frac{F_{n+2}}{F_{n+1}}-\frac{F_{n+3}}{F_{n+2}}\right)+\ldots+\left(\frac{F_m}{F_{m-1}}-\frac{F_{m+1}}{F_m}\right)\right|=$$

$$=\left|\frac{(F_{n+1})^2-F_{n+2}F_n}{F_nF_{n+1}}+\frac{(F_{n+2})^2-F_{n+3}F_{n+1}}{F_{n+1}F_{n+2}}+\ldots+\frac{(F_m)^2-F_{m+1}F_{m-1}}{F_{m-1}F_m}\right|=\left|\frac{(-1)^{n+2}}{F_nF_{n+1}}+\frac{(-1)^{n+3}}{F_{n+1}F_{n+2}}+\ldots+\frac{(-1)^{m+1}}{F_{m-1}F_m}\right|=$$

$$\overset{*}{=}\left|\frac{1}{F_nF_{n+1}}-\frac{1}{F_{n+1}F_{n+2}}+\ldots+(-1)^{m-n-1}\frac{1}{F_{m-1}F_m}\right|\leq\left|\frac{1}{F_nF_{n+1}}\right|\leq\frac{1}{F_p}\leq\frac{1}{p}<\varepsilon .$$

The sequence in question thus meets criterion (CBcc) and is therefore convergent.

Mark $\lim\limits_{n\to\infty}\frac{F_{n+1}}{F_n}=\varphi$. Then we obtain $\lim\limits_{n\to\infty}\frac{F_{n+1}}{F_n}=\lim\limits_{n\to\infty}\frac{F_n+F_{n-1}}{F_n}=\lim\limits_{n\to\infty}\left(1+\frac{F_{n-1}}{F_n}\right)$, implying $\varphi=1+\frac{1}{\varphi}$. The solution of quadratic equation $\varphi^2-\varphi-1=0$ includes numbers $\varphi_{1,2}=\frac{1\pm\sqrt{5}}{2}$. In view of the conditions of the task, the limit we were expected to determine is the constant of the golden section, i. e. $\varphi=\frac{1+\sqrt{5}}{2}$.

<u>Example 3.</u>

Find out whether the sequence defined by recursion $a_{n+1}=\frac{1}{2}\left(a_n+\frac{a}{a_n}\right)$, $a_1>0$, converges. ($a\geq 0$).

Solution. Obviously, for all $n\in\mathbb{N}$ it holds that $a_n>0$, i. e. the sequence is bounded below. We need to analyze the monotony of the sequence. Comparing two subsequent terms we obtain hypothesis that the given sequence is non-increasing. In order to prove the hypothesis, assume that $a_{n+1}\geq a_{n+2}$. Then

$$a_{n+1}\geq a_{n+2}\Leftrightarrow a_{n+1}\geq\frac{1}{2}\left(a_{n+1}+\frac{a}{a_{n+1}}\right)\Leftrightarrow$$

$$\Leftrightarrow 2(a_{n+1})^2\geq a+(a_{n+1})^2\Leftrightarrow(a_{n+1})^2\geq a\Leftrightarrow a_{n+1}\geq\sqrt{a} .$$

We have obtained an equivalent statement, which, however, must be also verified:

* We made use of the Cassini's identity, i. e. $F_{n-1}F_{n+1}-F_n^2=(-1)^n$.

$$a_{n+1} \geq \sqrt{a} \Leftrightarrow \frac{1}{2}\left(a_n + \frac{a}{a_n}\right) \geq \sqrt{a} \Leftrightarrow a + (a_n)^2 \geq 2a_n\sqrt{a} \Leftrightarrow \left(\sqrt{a} - a_n\right)^2 \geq 0 .$$

We can see that the given sequence is in fact non-increasing and bounded below, therefore according to theorem (*Wt2*) its limit exists. Mark $\lim_{n\to\infty} a_n = \xi$. Having applied limit transition in identity $a_{n+1} = \frac{1}{2}\left(a_n + \frac{a}{a_n}\right)$ we have $\xi = \frac{1}{2}\left(\xi + \frac{a}{\xi}\right)$, thus $\xi = \sqrt{a}$. The terms of the given sequence, thus, converge to the square root of number a, and this "approaching" is relatively fast. This can be demonstrated for value $a = 100$ and first estimate of $a_1 = 20$. Then, we obtain $a_2 = 12{,}5$; $a_3 = 10{,}25$; $a_4 = 10{,}00305\ldots$ At real approximation the first estimate can be, naturally, "more reasonable".

Example 4.

Kepler's equation $x = q\sin x + a$ is used for determination of the position of planets in their orbits; while $0 < q < 1$, $a \in \mathbb{R}$ are known constants, x is expected to be determined. Show that this equation has a single solution.

Solution. Let $x_0 \in \mathbb{R}$. Construct a number sequence $\{x_n\}_{n=1}^{\infty}$, where

$$x_1 = q\sin x_0 + a,\ x_2 = q\sin x_1 + a,\ \ldots,\ x_{n+1} = q\sin x_n + a, \text{ etc.}$$

Then $$x_2 - x_1 = q(\sin x_1 - \sin x_0) = 2q\sin\frac{x_1 - x_0}{2}\cos\frac{x_1 + x_0}{2},$$

which implies $$|x_2 - x_1| \leq q|x_1 - x_0|.$$

Similarly $|x_3 - x_2| \leq q|x_2 - x_1|$, or $|x_3 - x_2| \leq q^2|x_1 - x_0|$, i. e. $|x_{n+1} - x_n| \leq q^n|x_1 - x_0| \quad (*)$.

Now, let $m > n$, then $x_m - x_n = x_m - x_{m-1} + x_{m-1} - x_{m-2} + x_{m-2} - \ldots + x_{n+1} - x_n$.

Applying $(*)$ we obtain

$$|x_m - x_n| \leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \ldots + |x_{n+1} - x_n| \leq$$
$$\leq \left(q^{m-1} + q^{m-2} + \ldots + q^n\right)|x_1 - x_0| = q^n\frac{1 - q^{m-n}}{1 - q}|x_1 - x_0|.$$

Let $n \to \infty$, then for the individual factors it holds that $q^n \to 0$, $0 < \frac{1 - q^{m-n}}{1 - q} < 1$, $|x_1 - x_0|$ is a fixed number. It means that $|x_m - x_n| \to 0$, or in other words criterion (*CBcc*) is met, i. e. limit $\lim_{n\to\infty} x_n = \xi$ exists.

Applying limit transition in relation $x = q\sin x + a$ we obtain $\xi = q\sin\xi + a$, so number ξ is the solution to Kepler's equation. Now, we need to prove that this solution is independent of the choice of value x_0, which means that the position of the planet is univocal. Actually, if there were also $\vartheta \neq \xi$ for which it would hold

$\vartheta = q\sin\vartheta + a$, then $\xi - \vartheta = q\left(\sin\xi - \sin\vartheta\right) = \; = 2q\sin\dfrac{\xi-\vartheta}{2}\cos\dfrac{\xi+\vartheta}{2}$.

The latter suggests that $$|\,\xi-\vartheta\,| \le 2q\left|\sin\frac{\xi-\vartheta}{2}\right| \le q\,|\,\xi-\vartheta\,|.$$

With respect to the value of number q the inequality holds only if $|\,\xi-\vartheta\,| = 0$, i. e. $\xi = \vartheta$.

Conclusion

There are many techniques for computation of limits of sequences, depending on the type of the limit. However, in such cases the sequences are defined analytically. In the paper we present selected problem tasks – obstacles, non-standard nature of the circumstances and impossibility to use algorithmic methods to solve the tasks are caused by the fact that the sequences in question are defined by recursion. Firstly, as an example we chose a general mathematical task, secondly, a task whose solution is a well-known constant φ. Then we continued with a sequence whose terms approximate the square root of a positive number, and finally, we demonstrated an application of this issue in celestial mechanics.

References

[1] Fichtengoľc, G. M.: *Kurs differencialʹnogo i integraľnogo isčislenija I*. Moskva: Nauka, 1962. 607 s.

[2] Grebenča, M. K., Novoselov, S. I.: *Kurs matematičeskogo analiza 1*. Moskva: Učpedgiz, 1951. 543 s.

[3] Trench, W. F.: *Introduction to real analysis*. NJ: Pearson, 2003. 583 s. ISBN: 0-13-045786-8.

[4] Varga, M.: *Úvod do diferenciálneho počtu*. Nitra: UKF, 2015. 143 s. ISBN: 978-80-558-0868-0.