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On One Strategy of Solving Problem Tasks in Mathematics

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Abstract

The strategy of penultimate step (way back) is the important method that turns solution of a mathematical problem on target. The detection and the awakening of the immediate cause or the condition of a contrived state are essential at choice of mathematical aids of solving a problem. Then we concentrate on the probation of the state which precedes the verified state directly and it is its precondition. The possibilities usage of this strategy are analysed in various mathematical branches in the submitted contribution. The effectiveness of the method in question is illustrated via solutions of several miscellaneous mathematical problems.

Keywords: mathematical strategy, penultimate step, mathematical problem

Classification: 00A35; 97C50; 97D50

Introduction

The terminologies of mathematical branches are verified already by their histories and they are therefore conventional. The terminology of "theory" of solving problem tasks in mathematics is not accomplished comprehensively. It is mostly popularized by the term heuristics (Polya [3] 1948; Polya [4] 1954). On the other hand an experienced solver of mathematical problems works in several various levels that are artificial by usage of mathematical methods with different comprehension of application. Those most general are known as mathematical strategies usually (Zeitz [5] 1999; Kopka [1] 2004; Vrábel [4] 2005). The method *penultimate step (way back)* is one of these strategies. This strategy is based on the identification of the immediate preceding situation that induces the required (contrived) state. Then we do not concentrate on proving of the required state, but on proving the precondition which directly induces this state.

The strategy of *penultimate step*

In fact, while using the strategy of *penultimate step* we ask what condition implicates the contrived assertion (situation). In other words, we need to approach the problem a converse perspective. We can, thus, talk about "working backwards" too. If we have isolated the penultimate step, then we have reduced a problem into the simple deductive statement

$$\text{The truth of the penultimate step} \Longrightarrow \text{The conclusion.}$$

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Of course, establishing the penultimate step may involve various forms of argument. One of them can also have the logical scope $p_1 \Rightarrow p_2 \Rightarrow \cdots \Rightarrow p_n$. The well-known games Nim based on removing playing stones are typical examples on such situation.

Problem 1. Have n ($n \in \mathbb{N}$) playing stones on a pile. Two players take from the pile one, two or three stones alternately. The winner is the player who takes the last stone (stones). Investigate that the stating player has a winning strategy (the winning strategy is such manner of game of a player which secures him/her victory regardless the opponent´s manner of game).

Solution. Every natural number n can be expressed in form $n = 4q + r$, where $r \in \{0,1,2,3\}$. Working backwards we can easily understand that a player wins in case he moves, provided that 1, 2 or 3 stones are left in the pile. The player cannot move, if four stones are left on the pile. Similarly, he cannot move if 8 stones, 12 stones, … , $4q$ stones are on the pile because its opponent sets him to these critical (losing) positions. Thus, the starting player wins if $r \in \{1,2,3\}$. The starting player takes 3 stones at the beginning of the game. The opponent is in the losing position because $4q$ stones are in the pile. Subsequently, the starting player has the possibility to keep of playing in such a way that the opponent has to move from the situation when there are $4q$, $4(q-1)$, ... , 8 or 4 stones are in the pile. On other hand, the starting player loses (the opponent has the winning strategy), if there are $4q$ stones in the pile at the beginning of the game.

The following elementary situations are typical examples of applying the *penultimate step*:

- a proof of equality of two number expressions A , B (separately, we prove inequalities $A \leq B, B \leq A$ instead of directly proving equality $A = B$);
- a proof of equality of two sets X, Y (here, the penultimate step is based on proving inclusions $X \subseteq Y, Y \subseteq X$);
- a proof of an assertion in form of equivalence of two statements p, q (here, the penultimate step is based on proving implications $p \Rightarrow q, q \Rightarrow p$).

The identification of penultimate step is not always as simple and straightforward as in problem 1. Usually it is necessary to use also other mathematical methods in order to establish the penultimate step. For example consider the following task:

Prove that the product of four consecutive natural numbers cannot be the square of an integer.

We cannot think of any easy criteria for determining when a number cannot be a square of an integer. By experimentation, for $f(n)$, $f(n) = n(n+1)(n+2)(n+3)$, we have the following values:

$$f(1) = 24 = 5^2 - 1, f(2) = 120 = 11^2 - 1, f(3) = 360 = 19^2 - 1,$$

$$f(4) = 840 = 29^2 - 1, f(5) = 1680 = 41^2 - 1, f(20) = 212\,520 = 431^2 - 1$$

$$f(113) = 171\,845\,880 = 13109^2 - 1.$$

The hypothesis "$f(n)$ *is one less than a perfect square for any* n" is legitimate. Evidently, that $k^2 - 1$ ($k \in \mathbb{N}$) is not a square of an integer. We discover the penultimate step, and the new task is to prove that

"$f(n)$ *is the form* $k^2 - 1$ *for any* n", which is quite easy:

$$f(n) + 1 = (n^2 + 3n)(n^2 + 3n + 2) + 1 =$$
$$\big((n^2 + 3n + 1) - 1\big)\big((n^2 + 3n + 1) + 1\big) + 1 = (n^2 + 3n + 1)^2.$$

Applications of the p*enultimate* step strategy in problem solving

Problem 2. Investigate whether there are such natural numbers m, n for which it holds

(1) $$5^m - m^2 + 3(m + 1) = 2^n + 1$$

is valid.

Solution. Consider most likely that equation (1) does not have any solution in the set of natural numbers. Two number expressions X, Y with integral values do not equal besides when one of them is always an even number and the second one is always an odd number. Therefore, using the strategy of penultimate step we aim to prove that the number expressions $5^m - m^2 + 3(m + 1)$, $2^n + 1$ have a different parity. Number $2^n + 1$ is odd for any $n \in \mathbb{N}$. On the other hand, $5^m - m^2 + 3(m + 1)$ is an even number for any $m \in \mathbb{N}$. It is sufficient to consider cases for even and odd m particularly. If m is odd then the terms $5^m, m^2$are odd and $3(m + 1)$ is even. If m is even then the terms 5^m, $3(m + 1)$ are odd and m^2 is even.

Problem 3. Find the greatest common divisor of numbers a, b, if

$$a = 2^{2004} - 1, b = 2^{2001} - 1.$$

Solution. Numbers A, B or numbers $A, A - B$ have the same greatest common divisor. Therefore, the penultimate step is based on finding the greatest common divisor d of numbers $a, 7.\,2^{2001}$, which is easier. The number a is odd, therefore d is also odd. Number $7.\,2^{2001}$ has only odd natural divisors, namely numbers 1, 7. Number 7 divides also number a, because

$$a = (2^3 - 1)(2^{2001} + 2^{1998} + \cdots + 2^3 + 1).$$

This implies that 7 is the greatest common divisor of numbers a, b.

Remark. Number 7 divides also number b because

$$b = (2^3 - 1)(2^{1998} + 2^{1995} + \cdots + 2^3 + 1).$$

However, it is further necessary to prove that numbers $2^{2001} + 2^{\,1998} + \cdots + 2^3 + 1$, $2^{1998} + 2^{1995} + \cdots + 2^3 + 1$ are relatively prime.

Problem 4. Let M be an arbitrary subset of set $\{1,2, \dots, 2n\}$ with $n + 1$ elements. Prove that M contains two numbers which are relatively prime.

Solution. Numbers $k, k + 1$ are relatively prime for any natural number k. If a natural number d divides both numbers $k, k + 1$, then it divides also their difference $k + 1 - k = 1$, hence $d = 1$. In this case the penultimate step lies in proving that an arbitrary subset of set $\{1,2, \dots, 2n\}$ with $n + 1$ elements contains at least one couple of natural numbers $k, k + 1$, where $k + 1 \le 2n$. Indeed, this follows easily from the Dirichlet´s Box Principle (=pigeonhole principle) because

$$\{1,2, \dots, 2n\} = \{1,2\} \cup \{3,4\} \cup \cdots \cup \{2n - 1, 2n\}.$$

The sets which are to the right of the last equality are mutually disjoint and their number is n. Set M has $n+1$ elements, therefore two its elements necessarily belong to one of the sets $\{1,2\}, \{3,4\}, \dots, \{2n-1, 2n\}$.

Problem 5. Let $x_1, x_2, \dots, x_n$ be an arbitrary permutation of numbers $1, 2, \dots, n$. Prove that

$$(x_1+1)(x_2+2)\cdot \cdots \cdot (x_n+n)$$

is an even number for every odd natural number n.

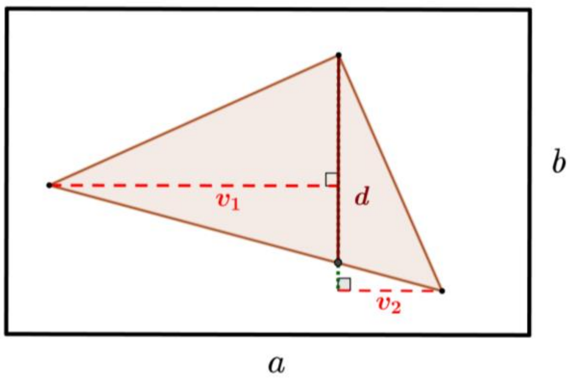
Solution. The product of several natural numbers is even if and only if at least one of the factors is an even number. We shall therefore prove that at least one of numbers x_1+1, $x_2+2, \dots, x_n+n$ is even. Compute the sum of these numbers:

$$(x_1+1)+(x_2+2)+\cdots+(x_n+n)=$$
$$(x_1+x_2+\cdots+x_n)+(1+2+\ldots+n)=2(1+2+\ldots+n).$$

This sum is, thus, an even number. If the sum odd quantity of natural numbers is even, then at least one of these numbers must be even.

Problem 6. Consider a square $10\ \mathrm{cm}\times 10\ \mathrm{cm}$. There are 101 points in this square, and there are no three points among them lying in a line. Show that there is a triplet of points among them which form a triangle with area no more than 1 cm^2.

Solution. The area of an arbitrary triangle inscribed in a rectangle with side a, b does not exceed $\frac{ab}{2}$. This is evident from the following figure.



It is possible to divide such a triangle into two triangles with a common side of length d which is parallel to one side of the rectangle (b). Apparently, $d \le b$. The sum of altitudes v_1, v_2 of both triangles is not greater than the length of the second side of the rectangles (a). The sum of the areas of the two triangles is equal to number $\frac{dv_1}{2}+\frac{dv_1}{2}$. We obtain

$$\frac{dv_1}{2}+\frac{dv_1}{2}=\frac{1}{2}d(v_1+v_2)\le\frac{1}{2}ab.$$

The penultimate step here is based on the possibility of dividing a given square to 50 rectangles with area 2 cm^2. At least three points from the given 101 points get onto at least one rectangle from these 50 rectangles. It is the consequence of the Dirichlet´s Box Principle. It is sufficient to divide one side of the square into 10 segments measuring 1 cm in length and the remaining side into 5 segments with length 2 cm. This way we obtain the net dividing the given square with 50 rectangles.

Problem 7. Prove that for every natural number n it holds that

$$\lfloor\sqrt{n}+\sqrt{n+1}\rfloor=\lfloor\sqrt{4n+2}\rfloor,$$

where $\lfloor x\rfloor$ denotes the floor of a real number x.

Solution. If there is an integer m with the property $a,b\in\langle m,m+1)$ for some real numbers a,b, then $\lfloor a\rfloor=\lfloor b\rfloor$. Hence, the use of the penultimate step here is based on proving that $\lfloor\sqrt{4n+2}\rfloor\in\langle k,k+1)$ for each natural number n, where $k=\lfloor\sqrt{n}+\sqrt{n+1}\rfloor$.

It is evident that inequalities

$$\sqrt{n}+\sqrt{n+1}<\sqrt{4n+2}<\sqrt{n}+\sqrt{n+1}+1$$

are true for any $n\in\mathbb{N}$. These inequalities are consequence of relations

$$\left(\sqrt{n}+\sqrt{n+1}\right)^2=2n+1+2\sqrt{n}\sqrt{n+1}<4n+2<$$

$$2n+2+2\sqrt{n}\sqrt{n+1}+2\sqrt{n}+2\sqrt{n+1}=\left(\sqrt{n}+\sqrt{n+1}\ +\ 1\right)^2.$$

If n is an arbitrary natural number, then $4n+2$ cannot be a square of an integer. The remainder after the division of natural number m^2 by number 4 is 0 or 1 for any $m\in\mathbb{N}$, but for number $4n+2$ equals 2. The equality $\lfloor x\rfloor=\lfloor x+1\rfloor$ is true for every $x\in\mathbb{R}$. Thus, we obtain

$$k^2=\lfloor\sqrt{n}+\sqrt{n+1}\rfloor^2<4n+2<\lfloor\left(\sqrt{n}+\sqrt{n+1}\right)+1\rfloor^2=(k+1)^2,$$

$$k<\sqrt{4n+2}<k+1,$$

$$k=\lfloor\sqrt{n}+\sqrt{n+1}\rfloor=\lfloor\sqrt{4n+2}\rfloor.$$

Conclusion

It is convenient to observe certain steps verified by practice when we solve problem tasks in mathematics. The consistent understanding and the accurate orientation are primary. The determination of a reasonable beginning strategy appertains to the initial steps when solving mathematical problems. We use mainly the following methods:

- solving a more general problem, or on the contrary, a more specific one;
- experimentation, testing particular cases, solving a easier problem with some changed or missing conditions;
- using the method and the result of a similar problem, reformulation and transformation of a problem;
- decomposition of the task solution to various separate cases;
- consideration of a possible estimation of a solution, using a graphical method and implementation of a suitable quotation;
- identification of an immediate occasion that induces a required (contrived) state, i.e. the penultimate step strategy.

The latter plays the most important role in the reasonable alignment of solving most of mathematical problems.

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