# Theoretical and Didactic Aspects of Numeration of Real Numbers 

Peter Vrábel<br>*Department of mathematics, Faculty of Natural Sciences, Constantine the Philosopher University in Nitra, Tr. A. Hlinku 1, SK-949 74 Nitra,

Received 4 October 2015; received in revised form 7 October 2015; accepted 9 October 2015


#### Abstract

The notion of number incorporates a few of aspects. It is mainly its sense accordingly its definition as a mathematical object and the assignation of fundamental properties of operations with numbers. The notation of numbers expressed by admissible characters and algorithms of operations are further aspects. Mathematically least important aspect is the verbal denomination of numbers finally. First aspect belongs to basic equipment of mathematicians and it has purely theoretical character. The theoretical bases of notation of numbers and also possibilities their interpretations in education of mathematics on grammar schools and in the preparation of teachers of mathematics are determined in the paper.


Keywords: natural and real numbers, Archimedean property, whole part of number, notation of number.
Classification: 97C70, 97C90, 97U20

## Introduction

The comprehension of the principle notation of natural and rational numbers (we omit the notion of real number) in positional numeration system belongs in intellectual outfit perhaps of every person. It is possible to form the notion of natural even rational number onto primary school intuitively. There is a deeper understanding of essence numeration of numbers in the first year-class of grammar school. However the work with numbers of pupils even students and peoples anywise taper to acquaintance of algorithms various operations. But already the importance of other numeration system as decimal for example binary hints that it should not be so. It appears at the moment through numeration of numbers that these algorithms of calculation are as good as equal in the positional numeration with any admissible ground. It is important for teachers of mathematics so as they perceived that basic knowledge make possible to inscribe positive numbers whether or not natural, rational or real effectively and moreover they guarantee comfortable practice of basic operations with numbers (Bürger [1] 1973; Smítal [2] 1977; Vilenkin [4] 1977).

## 1 Assumed knowledge numeration of numbers

The exact definition of numbers as mathematical objects is fundamental assignment of theoretical arithmetic and it belongs at the university. Natural numbers inclusive of zero ( $\mathbb{N}_{0}$ ) are defined either as cardinal numbers of finite sets with utilization of set equivalence or they are defined axiomatically by Peano's system of axioms. The structure of integers $\mathbb{Z}$ is raised consequently by algebraic middles over extension of the structure ( $\mathbb{N}_{0},+, \cdot$ ) and the structure

[^0]of rational numbers $\mathbb{Q}$ by extension of the structure $(\mathbb{Z},+, r)$. Real numbers $(\mathbb{R})$ are defined either through the medium Dedekind's cuts of ordered set $\mathbb{Q}$ or by fundamental sequences of rational numbers (Šalát [3], 1982). None of that is possible to use on grammar school, especially as regards natural numbers according as real numbers.
The idea of equivalent sets emerges beside each other one-one assigning of elements of two sets with the same finite number points. So we can for example number two to explicate as common property of sets that contain some element and still one element, that they are equivalent, hence there is between them mentioned one-one assigning of elements.
The key scopes for numeration of numbers are following theorem (assertions):
A. Every non-empty subset of the set $\mathbb{N}$ has the smallest element.
B. Let $a \in \mathbb{N}_{0}\left(\mathbb{Q}_{0}^{+}, \mathbb{R}_{0}^{+}\right), b \in \mathbb{N}\left(\mathbb{Q}^{+}, \mathbb{R}^{+}\right)$. Then there is such natural number $n$, that $a<$ $b n$ (Archimedean property).
It is possible to advance proof of these assertions also on grammar school:
Let $\emptyset \neq M \subseteq \mathbb{N}_{0}$. Take arbitrary element $m \in M$ and then smallest element of finite set $\{0,1, \ldots, m\} \cap M$ is also smallest element of set $M$.
We can consider in the case B as follows. Let $a, b \in \mathbb{Q}_{0}^{+}, a=\frac{p}{q^{\prime}}, b=\frac{r}{s^{\prime}}, p, q, r, s \in \mathbb{N}$. It suffices to put $n=p(s+1)$ because
$$
a=\frac{p}{q} \leq p<p \frac{(s+1)}{s}=\frac{1}{s} p(s+1) \leq \frac{r}{s} p(s+1)=b p(s+1) .
$$

Now let $a, b \in \mathbb{R}_{0}^{+}$. We could come from the apprehension that there are positive rational numbers $a_{1}, a_{2}, b_{1}, b_{2}$ such that $a_{1}<a<a_{2}, b_{1}<b<b_{2}$. There is natural number $n$ with property $a_{2}<b_{1} n$ on the ground of the preliminary consideration. Then $a<b n$.
The known theorem from number theory about division with remainder is accordingly the corollary of the assertions A and B :
Let $a \in \mathbb{N}_{0}$ and $b \in \mathbb{N}$. There are unique numbers $q, r \in \mathbb{N}_{0}$ such that

$$
a=b q+r, 0 \leq r<b .
$$

We outline the proof with the utilization of the theorem $\mathrm{A}, \mathrm{B}$. The set $M=\{n \in \mathbb{N} ; a<b n\}$ is non-empty subset of the set $\mathbb{N}$ on the ground of $B$. The set $M$ has the smallest element $u$ pursuant to A. Put $q=u-1$ and $r=a-b q$, so $a=b q+r$. The following relationships

$$
q b=(u-1) b \leq a<b u
$$

result from the inequality $u-1<u$ and from the choice of $u$.
Then we obtain that

$$
0 \leq r=a-b q<b u-(u-1) b=b .
$$

If $a=b q_{i}+r_{i}, 0 \leq r_{i}<b, i=1,2$, then $b\left|q_{1}-q_{2}\right|=\left|r_{2}-r_{1}\right|$. It is possible only in the case that $q_{1}=q_{2}, r_{1}=r_{2}$, because $\left|r_{2}-r_{1}\right|<b$.
If we apply the theorem about division with remainder repeatedly for $b>1$ so we obtain

$$
\begin{array}{ll}
a=b q_{0}+a_{0}, & 0 \leq a_{0}<b \\
q_{0}=b q_{1}+a_{1}, & 0 \leq a_{1}<b \\
\quad \vdots & \\
q_{n-2}=b q_{n-1}+a_{n-1}, & 0 \leq a_{n-1}<b
\end{array}
$$

$$
q_{n-1}=b .0+a_{n}, \quad 0 \leq a_{n}<b
$$

The sequence nonnegative whole numbers $q_{0}, q_{1}, \ldots$ is decreasing apparently, so there is such $n \in \mathbb{N}$ that $q_{n}=0$. Then

$$
a=b q_{0}+a_{0}=b^{2} q_{1}+b a_{1}+a_{0}=\cdots=b^{n} a_{n}+b^{n-1} a_{n-1}+\cdots+b a_{1}+a_{0}
$$

Instead $a=b^{n} a_{n}+b^{n-1} a_{n-1}+\cdots+b a_{1}+a_{0}$ we write briefly $a=\left(a_{n} a_{n-1} \ldots a_{1} a_{0}\right)_{b}$. We say also about number notation in the $b$-adic notation system.

The following assertion is further simple corollary of assertions $A, B$ which is due in number notation:

For every $x \in \mathbb{R}^{+}$there is unique such $m \in \mathbb{N}_{0}$ that

$$
m \leq x<m+1
$$

Actually if we put $b=1$ in Archimedean property then we obtain:
for every $x \in \mathbb{R}^{+}$there is such $n \in \mathbb{N}$ that $x<n$. The set $M_{1}=\{n \in \mathbb{N} ; x<n\} \subseteq \mathbb{N}$ has on the ground of $A$ the smallest element $u$. Then $u-1 \leq x<u$. Put $m=u-1$. The unicity of $m$ is evident.

The number $m$ is called whole part of real number $x$. We denote its by sign $[x]$.
It is needed to sense that for every number $x \in \mathbb{R}^{+}$there is $n \in \mathbb{N}$ with the property $\frac{1}{n}<x$. It suffices to put pre $a=\frac{1}{x}$ a $b=1$ in the assertion B .

The notion of mathematical induction and the construction of a sequence by mathematical induction is last needed scope for real number notation in positional numeration system.

It is possible to advance on grammar school intuitively the fact that the set $M \subseteq \mathbb{N}$ satisfying the properties

$$
\text { a) } 1 \in M \text { and b) } n \in M \Rightarrow n+1 \in M
$$

equals to whole set $\mathbb{N}$ :
Let $m \in \mathbb{N}$. We gain number $m$ from number 1 by means of addition ones after $m-1$ steps. The element $m$ belongs to set $M$ basically assumptions a) and b). Often this fact is approached by the medium of the dominoes effect metaphorically.

The mathematical induction is used moreover on the construction of a sequence $\left\{a_{n}\right\}_{n=1}^{\infty}$ with required properties:

It is framed first member $a_{1}$. Moreover it is established how we design member $a_{n+1}$ in general, if we have designed member $a_{n}(n \in \mathbb{N})$. Then the set $M=\left\{n \in \mathbb{N}\right.$; $a_{n}$ is designed $\}$ satisfies the properties a$), \mathrm{b})$. Then $M=\mathbb{N}$ and thus it is designed each member the sequence $\left\{a_{n}\right\}_{n=1}^{\infty}$.

## 2 Decimal expansion of real numbers

Now we can introduce the final theorem about notation of nonnegative real number in the decimal numeration system. We do it on the ground of the results of the part 1 . We utilize nevertheless admittedly besides the notation of nonnegative integers. It is possible to carry the proof of first theorem about the existence of an expansion of real number on grammar school. The second theorem about unicity of this expansion appertains on college or grammar school with advanced school teaching of mathematics.

Theorem 1. For every nonnegative real number $a$ there are $a_{0} \in \mathbb{N}_{0}$ and such infinite sequence of numbers $a_{1}, a_{2}, \ldots, a_{n}, \ldots$, where $a_{n} \in\{0,1, \ldots, 9\}, n=1,2, \ldots$, that the following inequalities

$$
0 \leq a-\left(a_{0}+\frac{a_{1}}{10}+\frac{a_{2}}{10^{2}}+\cdots+\frac{a_{n}}{10^{n}}\right)<\frac{1}{10^{n}}
$$

are valid for any $n \in \mathbb{N}_{0}$.
The inequality $a_{n}<9$ is true for infinite number of $n$ ' $s$ additionally.
Proof. Let $a \in \mathbb{R}_{0}^{+}$. Put $a_{0}=[a], b_{1}=a-a_{0}$. Then

$$
a=a_{0}+b_{1}, a_{0} \in \mathbb{N}_{0}, 0 \leq b_{1}<1
$$

Denote $a_{1}=\left[10 b_{1}\right], b_{2}=10 b_{1}-a_{1}$. Then

$$
b_{1}=\frac{a_{1}}{10}+\frac{b_{2}}{10^{\prime}}, 0 \leq b_{2}<1,0 \leq 10 b_{1}<10 .
$$

Thus $a_{1} \in\{0,1, \ldots, 9\}, a=a_{0}+\frac{a_{1}}{10}+\frac{b_{2}}{10}$. If we designed numbers $a_{0}, a_{1}, \ldots, a_{n-1}$ and numbers $b_{1}, b_{2}, \ldots, b_{n}$ then we put $a_{n}=\left[10 b_{n}\right], b_{n+1}=10 b_{n}-a_{n}$. Then relations

$$
\begin{equation*}
a=a_{0}+\frac{a_{1}}{10}+\frac{a_{2}}{10^{2}}+\cdots+\frac{a_{n}}{10^{n}}+\frac{b_{n+1}}{10^{n}}, 0 \leq b_{n+1}<1 \tag{1}
\end{equation*}
$$

are true. By the mathematical induction we devised the sequence of numbers $a_{1}, a_{2}, \ldots, a_{n}, \ldots$ (besides the sequence $b_{1}, b_{2}, \ldots, b_{n}, \ldots$ ) with required properties. The inequalities

$$
0 \leq a-\left(a_{0}+\frac{a_{1}}{10}+\frac{a_{2}}{10^{2}}+\cdots+\frac{a_{n}}{10^{n}}\right)=\frac{b_{n+1}}{10^{n}}<\frac{1}{10^{n}}
$$

are valid also.
Now we prove that inequality $a_{n}<9$ is valid for infinite amount natural numbers $n$. We shall proceed indirectly. Let $m$ be such natural number that $a_{k}=9$ for any $k>m$. There is such $l \in \mathbb{N}$ that $1-b_{n+1}>\frac{1}{l}$. Evidently $\frac{1}{l}>\frac{1}{1^{l}}$. Hence we obtain

$$
\begin{equation*}
\frac{\mathrm{b}_{\mathrm{m}+1}}{10^{m+1}}<\frac{1}{10^{m}}\left(1-\frac{1}{10^{l}}\right)=\frac{9}{10^{m+1}}+\frac{9}{10^{m+2}}+\cdots \frac{9}{10^{m+l}} . \tag{2}
\end{equation*}
$$

Then from valid attitudes

$$
\begin{gathered}
a_{0}+\frac{a_{1}}{10}+\frac{a_{2}}{10^{2}}+\cdots+\frac{a_{m}}{10^{m}}+\frac{9}{10^{m+1}}+\frac{9}{10^{m+2}}+\cdots \frac{9}{10^{m+l}} \leq x= \\
a_{0}+\frac{a_{1}}{10}+\frac{a_{2}}{10^{2}}+\cdots+\frac{a_{m}}{10^{m}}+\frac{b_{m+1}}{10^{m+1}}
\end{gathered}
$$

we receive the opposite inequality to inequality (1). It is misunderstanding.
Theorem 2. Each nonnegative real number $a$ can be uniquely expressed in the form

$$
\begin{equation*}
a=a_{0}+\sum_{n=1}^{\infty} \frac{a_{n}}{10^{n}} \tag{3}
\end{equation*}
$$

where $a_{0} \in \mathbb{N}_{0}, a_{n} \in\{0,1, \ldots, 9\}, n=1,2, \ldots$, and for infinite number $n$ 's we have $a_{n}<9$.
Proof. We utilize the equality (1) and the designed sequences $\left\{a_{n}\right\}_{n=0}^{\infty},\left\{b_{n}\right\}_{n=1}^{\infty}$ in the theorem 1 with given properties. Denote

$$
s_{n}=a_{0}+\frac{a_{1}}{10}+\frac{a_{2}}{10^{2}}+\cdots+\frac{a_{n}}{10^{n}}, n \in \mathbb{N} .
$$

Then $\lim _{n \rightarrow \infty}\left(a-s_{n}\right)=0$ because $0 \leq a-s_{n}=\frac{b_{n+1}}{10^{n}}<\frac{1}{10^{n}}$. Hence we have $\lim _{n \rightarrow \infty} s_{n}=a$. The equality (3) emerges from the definition sum of infinite series. It is necessary to prove the unicity of the expression (3). Let be valid also the equality

$$
a=a_{0}^{\prime}+\sum_{n=1}^{\infty} \frac{a_{n}^{\prime}}{10^{n}}, a_{0}^{\prime} \in \mathbb{N}_{0}, a_{n}^{\prime} \in\{0,1, \ldots, 9\}, n=1,2, \ldots
$$

and let for infinite number $n^{\prime} s$ be valid $a_{n}^{\prime}<9$. Then

$$
\begin{gathered}
0 \leq c=\sum_{n=1}^{\infty} \frac{a_{n}}{10^{n}}<1,0 \leq c^{\prime}=\sum_{n=1}^{\infty} \frac{a_{n}^{\prime}}{10^{n}}<1, \\
\left|a_{0}-a_{0}^{\prime}\right|=\left|c-c^{\prime}\right|<1 .
\end{gathered}
$$

Number $\left|a_{0}-a_{0}^{\prime}\right|$ is whole and nonnegative. Thus $\left|a_{0}-a_{0}^{\prime}\right|=0$ and $a_{0}=a_{0}^{\prime}$. Then following equalities are valid:

$$
c=c^{\prime}, a_{1}+\frac{a_{2}}{10}+\cdots+\frac{a_{n}}{10^{n-1}}+\cdots=10 c=10 c^{\prime}=a_{1}^{\prime}+\frac{a_{2}^{\prime}}{10}+\cdots+\frac{a_{n}^{\prime}}{10^{n-1}}+\cdots .
$$

We can receive equality $a_{1}=a_{1}^{\prime}$ by similar consideration as for $a_{0}, a_{0}^{\prime}$. It can be already proved that $a_{n}=a_{n}^{\prime}$ for any $n \in \mathbb{N}_{0}$ by the mathematical induction. We inscribe the equality (3) in form

$$
\begin{equation*}
a=a_{0}, a_{1} a_{2} \ldots a_{n} \cdots, \tag{4}
\end{equation*}
$$

whereby integer $a_{0}$ is expressed in decimal numeration system.
The equality (4) means in fact that it is needed to add to nonnegative integer $a_{0}$ sum of infinite numerical series $\sum_{n=1}^{\infty} \frac{a_{n}}{10^{n}}$. The right side of the equality (4) is numeration of nonnegative real number $a$ in the decimal positional system. It is called also the decimal (decadic) expansion of the number $a$. The numbers $a_{n}(n=1,2, \ldots)$ are called decimal (decadic) numeral (figure, digit).

If is the decimal expansion of a number $a$ periodic accordingly

$$
a=a_{0}, c_{1} c_{2} \ldots c_{p} d_{1} d_{2} \ldots d_{k} \ldots d_{1} d_{2} \ldots d_{k} \ldots, c_{i}, d_{j} \in\{0,1, \ldots, 9\}, i=1, \ldots p, j=1, \ldots, k
$$

we obtain

$$
\begin{gathered}
a=a_{0}+\frac{c_{1}}{10}+\frac{c_{2}}{10^{2}}+\cdots \frac{c_{k}}{10^{k}}+\frac{u}{10^{p}}+\frac{u}{10^{p+k}}+\cdots= \\
a_{0}+\frac{c_{1}}{10}+\frac{c_{2}}{10^{2}}+\cdots \frac{c_{k}}{10^{k}}+\frac{u}{10^{p}} \frac{1}{1-\frac{1}{10^{k}}}=a_{0}+\frac{c_{1}}{10}+\frac{c_{2}}{10^{2}}+\cdots \frac{c_{k}}{10^{k}}+u \frac{10^{k-p}}{10^{k}-1} .
\end{gathered}
$$

where $u=\frac{d_{1}}{10}+\frac{d_{2}}{10^{2}}+\cdots \frac{d_{k}}{10^{k}}$.
We proved so that if the decimal expansion of a number $a$ is periodic then this number $a$ is rational.
It is valid also the inverted implication. Let $a=\frac{p}{q^{\prime}} p, q \in \mathbb{N}$. If we divide number $p$ with number $q$ then we can obtain partial remainders just from the set $\{0,1, \ldots, q-1\}$. It is possible only two cases:
we obtain the remainder 0 past finite number steps or a certain grouping of nonzero remainders is still repeated.
The number $a$ is rational in both cases.

## Conclusion

It is manipulated directly with decimal expansion of rational numbers only on the grammar school. We outlined possibilities approach of decimal expansion of real numbers on grammar school in the contribution. We integrated theorem 2 into schooling on college by reason of employment of the limiting process for sequences. On the other hand it makes merely use of convergent geometric series. The geometric series belong on grammar school.

## References

[1] Bürger, H., Schweiger, F. Zur Einführung der reellen Zahlen. Didaktik für Mathematik, 1973, 98-108.
[2] Smítal, J., Šalát, T. Reálne čísla (učebný text pre 3. ročník gymnázií s rozšíreným vyučovaním matematiky). Bratislava: SPN, 1977.
[3] Šalát, T. Reálne čísla. Bratislava: Alfa, 1982.
[4] Vilenkin, N.J. a kol. Matematika. Moskva: Prosveščenie, 1977.


[^0]:    * Corresponding author: pvrabel@ukf.sk

    DOI: 10.17846/AMN.2015.1.2.15-20

