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Selected Topics in the Extremal Graph Theory

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Abstract

Extremal problems in graph theory form a very wide research area. We study the following topics: the metric dimension of circulant graphs, the Wiener index of trees of given diameter, and the degree-diameter problem for Cayley graphs. All three topics are connected to the study of distances in graphs. We give a short survey on the topics and present several new results.

Keywords: Extremal graph theory, metric dimension, Wiener index, diameter, Cayley graph.

Classification: 05C35, 05C12

Introduction

Extremal graph theory is a sub-discipline of discrete mathematics focusing on identification and determination of maximal and minimal elements in partially ordered sets defined on, or arising from, specified families of graphs. Such type of analysis frequently reduces, in one way or another, to comparison of numerical parameters such as order of a graph (number of vertices), size (number of edges), diameter, degree, and so forth. More complex and important partial orders include graph minor relations, or containment as a subgraph (which may be required to be induced, topological, etc.). In fact, most results in the theory of finite graphs have been proved by considering extremal cases with respect to some partial order defined on graphs or on their parameter sets. In this sense, extremal graph theory plays a central role in the progress achieved in graph theory and, to some extent, also in the development of other branches of discrete mathematics.

We study the following problems:

- Metric dimension of circulant graphs,
- Wiener index of trees of given diameter,
- Degree-diameter problem for Cayley graphs.

Let G be a connected graph without loops and multiple edges. The distance d(u, v) between two vertices u, v in a graph G is the number of edges in a shortest path between them. The diameter of G is the greatest distance between all pairs of vertices of G. The degree of a vertex v is the number of edges incident to v. A tree is a graph, which does not contain cycles.

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Metric dimension of circulant graphs

We consider the following problem. A robot moves in a space, which is modelled by a graph. The robot moves from a node to a node, and it can locate itself by the presence of distinctively labelled landmark nodes. The position of robot is represented by its distances to a set of landmarks. The problem is to find the minimum number of landmarks required, and to find out where they should be placed, such that the robot can always determine its location. In graph theory language, a minimum set of landmarks which uniquely determine the position of robot is called a metric basis, and the minimum number of landmarks is called the metric dimension.

A set of vertices W is a resolving set of a graph G if every two vertices of G have distinct representations of distances with respect to W. The number of vertices in a smallest resolving set is called the metric dimension and it is denoted by dim(G). A connected graph G has dim(G) = 1 if and only if G is a path. Cycles have metric dimension 2.

We study the metric dimension of circulant graphs. For $m \le n/2$ the circulant graph $C_n(1, 2, ..., m)$ consists of the vertices $v_0, v_1, \dots, v_{n-1}$ and the edges $v_iv_{i+1}, v_iv_{i+2}, \dots, v_iv_{i+m}$, where $0 \le i \le n-1$ and the indices are taken modulo n.

The metric dimension of circulant graphs has been extensively studied. Javaid, Rahim and Ali [6] showed that

$$\dim(C_n(1, 2)) = 3 \text{ if } n \equiv 0, 2, 3 \pmod 4, \text{ and } \dim(C_n(1, 2)) \le 4 \text{ if } n \equiv 1 \pmod 4.$$

Imran et. al. [5] proved that for any $n \ge 12$,

$$\dim(C_n(1, 2, 3)) = 4 \text{ if } n \equiv 2, 3, 4, 5 \pmod 6,$$

$$\dim(C_n(1, 2, 3)) \le 5 \text{ if } n \equiv 0 \pmod 6, \text{ and } \dim(C_n(1, 2, 3)) \le 6 \text{ if } n \equiv 1 \pmod 6.$$

We extend known results on the metric dimension of circulant graphs by presenting bounds on the metric dimension of circulant graphs with 4 generators.

Theorem 1. For any $n \ge 21$,

$$4 \le \dim(C_n(1, 2, 3, 4)) \le 6.$$

Finally we state two open problems. The first one is to find exact values of the metric dimension of $C_n(1, 2)$, $C_n(1, 2, 3)$ and $C_n(1, 2, 3, 4)$ for any n, and the second one is to find upper and lower bounds on $C_n(1, 2, \dots, m)$ for $m \ge 5$.

Wiener index of trees of given diameter

The Wiener index is the oldest graph index. It has been investigated in the mathematical, chemical and computer science literature since the 1940's. With Wiener's discovery of a close correlation between the boiling points of certain alkanes and the sum of the distances between vertices in graphs representing their molecular structures, it became apparent that graph indices can potentially be used to predict properties of chemical compounds.

The Wiener index W(G) of a connected graph G is defined as the sum of the distances between all unordered pairs of vertices. The minimum value of the Wiener index of a graph (of a tree) of given order is attained by the complete graph (by the star), and the maximum value is attained by the path. It is not difficult to show that the extremal tree, which has the minimum Wiener index among trees of order n and diameter d, is the path of length d (containing d + 1 vertices) with the central vertex joined to the other n – d – 1 vertices; see [13].

The problem of finding an upper bound on the Wiener index of a tree (or graph) in terms of order and diameter is quite challenging; it was addressed by Plesník [11] in 1975, and restated by DeLaVina and Waller [3], but still remains unresolved to this date. We give a starting point to solving this long-standing problem. We present upper bounds on the Wiener index of trees of order n and diameter at most 6.

Theorem 2. Let T be a tree having n vertices and diameter d. Then the Wiener index of T is at most

(i) $5n^2/4 - 3n + 3$ if $d = 3$,

(ii) $2n^2 - 2n\sqrt{n-1} - 3n + 2\sqrt{n-1} + 1$ if $d = 4$,

(iii) $9n^2/4 - 2n^{3/2} + O(n)$ if $d = 5$,

(iv) $3n^2 - 2\sqrt{6}n^{3/2} - 2n + O(n^{1/2})$ if $d = 6$,

and the bounds are best possible.

The proof of this result can be found in [10]. The only tree of order n and diameter 2 is the star Sn having n – 1 leaves. Since any two leaves of the star are at distance 2, and the distance between the central vertex and any leaf is 1, the Wiener index of Sn is $n^2 - 2n + 1$. Let us mention that to find a sharp upper bound on the Wiener index for trees of given order and large diameter remains an open problem.

Note that there are indices which were introduced much later than the Wiener index, however upper bounds on these indices for trees of given order and diameter are known. For example, a sharp upper bound on the eccentric connectivity index of trees of given order and diameter was given in [9].

Degree-diameter problem for Cayley graphs

Suppose that one wants to set up a network in which each node has just a limited number of direct connections to other nodes, and one requires that any two nodes can communicate by a route of limited length. What is the maximum number of nodes one can have under the two constraints? It is clear that this question can be translated into the language of graph theory. The problem is to find the largest possible number of vertices in a graph of given maximum degree and diameter. Vertices of a graph represent nodes of a network, while edges represent connections.

A Cayley graph C(S, X) is specified by a group S and a unit-free generating set X for this group such that $X = X^{-1}$. The vertices of C(S, X) are the elements of S and there is an edge between two vertices u and v in C(S, X) if and only if there is a generator a in X such that v = ua.

The degree-diameter problem for Cayley graphs is to determine the largest number of vertices in a Cayley graph of given degree and diameter. Let $C_{d,k}$ be the largest order of a Cayley graph of degree d and diameter k. The number of vertices in a graph of maximum degree d and diameter k cannot exceed the Moore bound

$$M_{d,k} = 1 + d + d(d-1) + ... + d(d-1)^{k-1}.$$

In [1] Bannai and Ito improved the upper bound and showed that for any d, $k \geq 3$ there are no graphs of order greater than $M_{d,k} - 2$, therefore $C_{d,k} \leq M_{d,k} - 2$ for such d and k. Since the Moore graphs of diameter 2 and degree 3 or 7, and the potential Moore graph(s) of diameter 2 and degree 57 are non-Cayley (see [2]), Cayley graphs of order equal to the Moore bound exist only in the trivial cases when $d = 2$ or $k = 1$.

We focus on constructions of Cayley graphs of small diameter. Macbeth et al. [8] presented large Cayley graphs giving the bound $C_{d,k} \geq k((d-3)/3\}^k$ for any diameter $k \geq 3$ and degree $d \geq 5$. Let us also mention the Faber-Moore-Chen graphs [4] of odd degree $d \geq 5$, diameter k, such that $3 \leq k \leq (d+1)/2$, and order $((d+3)/2)! / ((d+3)/2 - k)!$. These graphs are vertex-transitive and in [8] it is proved that for any $k \geq 4$ and sufficiently large d the Faber-Moore-Chen graphs are not Cayley. Large Cayley graphs of given degree d and diameter k, where both d and k are small, were obtained by use of computers, see [7]. We state our results.

Theorem 3.

(i) $C_{d,3} \geq 3d^3/16$ for $d \geq 8$ such that d is a multiple of 4,

(ii) $C_{d,4} \geq 32(d/5)^4$ for $d \geq 10$ such that d is a multiple of 5,

(iii) $C_{d,5} \geq 25(d/4)^5$ for $d \geq 8$ such that d is a multiple of 4.

Proof. (i) Let H be a group of order $m \geq 2$ with unit element e. We denote by H^3 the product $H \times H \times H$. Let A be the automorphism of the group H^3, such that $A(x_1, x_2, x_3) = (x_3, x_1, x_2)$. We use the semidirect product S of H^3 and Z_{12} (the cyclic group with 12 elements) with multiplication given by

$$(x, y)(x', y') = (x\,A^y(x'), y + y'),$$

where A^y is the composition of A with itself y times, $x, x' \in H^3$ and $y, y' \in Z_{12}$. Elements of S will be written in the form $(x_1, x_2, x_3; y)$, where $x_1, x_2, x_3 \in H$ and $y \in Z_{12}$.

The generating set $X = \{ a_g, \bar{a}_{g'}, b_h, \bar{b}_{h'} \mid \text{for any } g, g', h, h' \in H\}$ where

$$a_g = (g, g, e; 1),\ \bar{a}_{g'} = (g', e, g'; -1),\ b_h = (h, e, e; 8) \text{ and } \bar{b}_{h'} = (e, h', e; 4).$$

It is easy to check that $X = X^{-1}$. The Cayley graph C(S, X) is of degree $d = |X| = 4m$ where $m \geq 2$ and order $|S| = 12m^3 = 12(d/4)^3 = 3d^3/16$.

We show that the diameter of C(S, X) is at most 3, which is equivalent to showing that each element of S can be obtained as a product of at most 3 generators of X. For any $x_1, x_2, x_3 \in H$ we have

$(x_1, x_2, x_3; 0) = (x_1, e, e; 8)(x_3, e, e; 8)(x_2, e, e; 8)$,

$(x_1, x_2, x_3; 1) = (x_1x_3^{-1}, e, e; 8)(x_3, x_3, e; 1)(e, x_2, e; 4)$,

$(x_1, x_2, x_3; 2) = (x_1, e, x_1; -1)(x_2, e, x_2; -1)(e, x_2^{-1}x_1^{-1}x_3, e; 4)$,

$(x_1, x_2, x_3; 3) = (x_1x_2^{-1}, e, e; 8)(x_3, e, e; 8)(x_2, e, x_2; -1)$,

$(x_1, x_2, x_3; 4) = (x_3, e, x_3; -1)(e, x_3^{-1}x_1x_2^{-1}, e; 4)(x_2, x_2, e; 1)$,

$(x_1, x_2, x_3; 5) = (x_1, e, e; 8)(x_3x_2^{-1}, e, e; 8)(x_2, x_2, e; 1)$,

$(x_1, x_2, x_3; 6) = (x_2x_3^{-1}, x_2x_3^{-1}, e; 1)(x_3, x_3, e; 1)(e, x_3x_2^{-1}x_1, e; 4)$.

It is easy to see that if $(x_1, x_2, x_3; y) = abc$, where $a, b, c \in X$ and $0 \le y \le 6$, then

$$(x_{y \pmod 3) + 1}^{-1}, x_{y+1 \pmod 3) + 1}^{-1}, x_{y+2 \pmod 3) + 1}^{-1}; -y) = c^{-1}b^{-1}a^{-1}.$$

Note that the diameter of C(S, X) cannot be smaller than 3, because the order of C(S, X) is greater than the Moore bound for diameter 2.

Proofs of (ii) and (iii) use similar techniques. □

By adding a few new elements to the generating sets, we get Cayley graphs of any degree $d \ge 10$ if $k = 4$, and any degree $d \ge 8$ if $k = 3$ or 5 (see [12]).

Corollary 1.

(i) $C_{d,3} \ge 3(d-3)^3/16$ for any $d \ge 8$,

(ii) $C_{d,4} \ge 32((d-8)/5)^4$ for any $d \ge 10$,

(iii) $C_{d,5} \ge 25((d-7)/4)^5$ for any $d \ge 8$.

These results improve the bounds of [8]. Particularly for diameter 3 we improve the lower bound considerably. It can be easily checked that the graphs of Faber, Moore and Chen are smaller than our graphs for diameter 3 and large degree, and they are larger than our graphs for diameters 4 and 5. However, for $k = 4$ and $d \ge 21$, and for $k = 5$ and $d \ge 23$, the Faber-Moore-Chen graphs are non-Cayley. To the best of our knowledge, for sufficiently large d there is no construction of Cayley graphs of degree d and diameter 3, 4 or 5 of order greater than the order of our graphs.

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