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Infinite Series in Physics

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Abstract

In calculus lectures and seminars students frequently call for demonstrations of usefulness of a new term or proposition. Tasks about tangent or instantaneous velocity are quite legendary, showing the need for derivatives. Similarly, pictures of inscribed or circumscribed rectangles, leading to the definition of Riemann definite integral, are quite popular. In the submitted contribution we aim to demonstrate the usefulness of infinite functional series (power, Taylor), which come in useful for common calculations in physics, and also for formulation of new discoveries and theories in physics.

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Classification: M15

Introduction

In mathematics classes with cast iron regularity teachers encounter the most favourite student question – "what is it good for?" And, this does not depend on the level of education, whether at elementary or secondary schools, or at universities. A possible response (well, difficult to say if satisfactory for the questioner) is introduction of selected application tasks. In the following lines we introduce several applications of infinite (functional) series in physics.

Position of a body

Let us assume that the position of a mass point in time t_0 is expressed by coordinate $x(t_0)$. If this point is moving, we are interested in its position after certain period of time Δt passes. If it is a „sufficiently short" time interval, we can consider it to be a case of uniform motion – i. e. the velocity $v = \frac{dx}{dt}$ was constant. Then, we get

$$x(t_0 + \Delta t) = x(t_0) + v \cdot \Delta t .$$

In real conditions, however, uniform motion of a body is quite rare. A more accurate estimate of the new position can be, thus, obtained if we assume that it was a case of uniformly accelerated motion with constant acceleration $a = \frac{d^2x}{dt^2}$.

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For the new position of the mass point we have

$$x\left(t_0+\Delta t\right)=x\left(t_0\right)+v\cdot\Delta t+\frac{1}{2}a\left(\Delta t\right)^2 .$$

Next, we can say that also the uniformly accelerated motion is quite a specific case, and the acceleration is not necessarily constant; there can be a kind of „acceleration of acceleration“. Let us label this quantity with symbol α; then $\alpha=\frac{d^3x}{dt^3}$. For the new position of the mass point we would get even more accurate value

$$x\left(t_0+\Delta t\right)=x\left(t_0\right)+v\cdot\Delta t+\frac{1}{2}a\left(\Delta t\right)^2+\frac{1}{6}\alpha\left(\Delta t\right)^3 .$$

The preceding assumptions imply that after infinitely many “improvements” the real position of the mass point would be

$$x\left(t_0+\Delta t\right)=x\left(t_0\right)+k_1\cdot\Delta t+k_2\left(\Delta t\right)^2+k_3\left(\Delta t\right)^3+\ldots;$$

where for each coefficient k_n applies

$$k_n=\frac{d^nx}{dt^n};\ \text{where}\ \ n=1,2,3\ldots$$

Finally, we can write

$$x\left(t_0+\Delta t\right)=x\left(t_0\right)+\left.\frac{dx}{dt}\right|_{t=t_0}\cdot\Delta t+\left.\frac{d^2x}{dt^2}\right|_{t=t_0}\cdot\left(\Delta t\right)^2+\left.\frac{d^3x}{dt^3}\right|_{t=t_0}\cdot\left(\Delta t\right)^3+\ldots .$$

From the mathematical point of view, we can say that function $x\left(t\right)$ is written in form of a power series, or in form of a Taylor series with midpoint t_0. The natural part of the mathematical solution would be determining the domain of convergence of the series, i. e. the set of all values t for which the sum of the series is a real number.

Magnetic force

Let there be an electric conductor, with current density τ. Recall that the charge of the conductor is determined by positive (protons) and negative particles (electrons), therefore $\tau=\tau^{+}+\tau^{-}$. If no electric current flows in the conductor, its total charge is zero, i.e. the body is electrically neutral. If current starts flowing in the conductor, its positive charge does not move, i. e. positive current density preserves the initial value, so $\tau^{+}=\tau_0$. However, electrons start moving within the conductor, and in terms of the theory of special relativity for negative current density we get $\tau^{-}=-\tau_0\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}$, where v is the velocity of electrons and c is the speed of light.

Here, mathematics enters the game, in form of the power (Taylor) series expansion for a function, as it holds good that

$$(1+x)^m = 1+\binom{m}{1}x+\binom{m}{2}x^2+\dots,$$

where $|x|<1$. In our case this requirement is met, as $\frac{v}{c}<1$, since nothing moves faster than light in vacuum. Moreover, already the third and all the following terms of the series are insignificantly small in comparison with the first and the second terms, therefore we can effectively work with the approximation

$$(1-x)^{-\frac{1}{2}} \approx 1+\frac{1}{2}x.$$

The total current density in the conductor is then

$$\tau = \tau^+ + \tau^- = \tau_0 - \tau_0\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} = \tau_0\left[1-\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}\right] \approx -\frac{\tau_0}{2}\cdot\frac{v^2}{c^2}.$$

Thus, the physical root of the situation can be handled in an "accelerated manner" – the force exerted by the field, which is formed by the moving electrons, on the charge with magnitude Q positioned in distance r, is

$$F_m = -QE = -Q\frac{\tau}{2\pi\varepsilon r} \approx Q\frac{\tau_0 v}{4\pi\varepsilon rc^2}v = QvB;$$

where B is the magnetic induction. Force F_m, which is actually the relativistic consequence of the existence of the electric force, is referred to as the magnetic force.

Einstein's equation

According to the theory of special relativity, the mass m of a body is not a constant quantity, but its value depends on the velocity v of the body. Naturally, this is only recognizable if the velocity is comparable with speed of light in vacuum c. If the rest mass of the body ($v = 0\, m/s$) is m_0, then the mass of the body moving at speed v is

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}.$$

Here, we encounter the series from the above lines again, namely

$$(1+x)^m = \sum_{n=0}^{\infty}\binom{m}{n}x^n = 1+\binom{m}{1}x+\binom{m}{2}x^2+\dots = 1+mx+\frac{m(m-1)}{2}x^2+\dots.$$

Then, we get

$$(1-x)^{-\frac{1}{2}} = 1+\frac{1}{2}x+\frac{3}{8}x^2+\dots, \text{ resp. } \left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} = 1+\frac{1}{2}\frac{v^2}{c^2}+\frac{3}{8}\frac{v^4}{c^4}+\dots.$$

As $v<c$, suffice it to use approximation $\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} \approx 1+\frac{1}{2}\frac{v^2}{c^2}$, leading to

$$m = m_0\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} \approx m_0\left(1+\frac{1}{2}\frac{v^2}{c^2}\right).$$

Having multiplied by c^2 we get

$$mc^2 = m_0c^2+\frac{1}{2}m_0v^2 = E_0+E_k.$$

On the right side there is the sum of rest and kinetic energy of the body, i. e. its total energy E, and thus we get the famous Einstein's equation

$$E = mc^2.$$

Conclusion

Mathematics is the queen of all sciences. Leonardo da Vinci even dared to say that if an exploration does not use mathematical methods, it cannot be considered a science. Applications of mathematics are surely present in many branches, perhaps the most appreciably in physics. Issues of functional series, power series, and Taylor series do not belong to introductory topics within university calculus courses, they are quite demanding. Similarly, their uses in physics (although we tried to minimize the physical stuff in this contribution) do not belong to the easiest examples of applications. If nothing more, we at least managed to emphasize that the birth of several spectacular physical theories, such as description of magnetism, or theory of relativity, was accompanied by these mathematical theories.

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