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Notes on Pythagorean Tetrahedron

Dušan Vallo[a*] – Viliam Ďuriš[a]

[a] *Department of Mathematics, Faculty of Natural Sciences, Constantine the Philosopher University in Nitra, Tr. A. Hlinku 1, SK - 949 74 Nitra*

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Abstract

Problem of existence of two classes of Pythagorean tetrahedrons is discussed in this paper. We derive characteristic equations of these tetrahedrons and we put the equations to the computer´s tests. Some models of the Pythagorean tetrahedrons are computing in this paper, too.

Keywords: Pythagorean Theorem, Pythagorean triple, tetrahedron, prism, characteristic tetrahedron.

Classification: G45

Introduction

Tetrahedron is a simplex in three dimensional space and such one is in some respects a more natural analogue for the triangle in two dimensional space. We consider a tetrahedron which edges form a Pythagorean triples representing three integer side lengths of right triangles as faces of the tetrahedron. [1] This point of view we develop in two ways.

At first, in according to [2] we re-investigate the existence of Pythagorean tetrahedrons which planar angles at one vertex are right. This type of the tetrahedron we name such Pythagorean tetrahedron of the 1st type.

One is called tetrahedron of the 2nd type if its planar angles at two different vertices are right. We also derive some characteristic equation with analogous parameters which determine an existence of tetrahedron of the 2nd type.

Note. The tetrahedrons of the 2nd type play a fundamental role in space filling. [3]

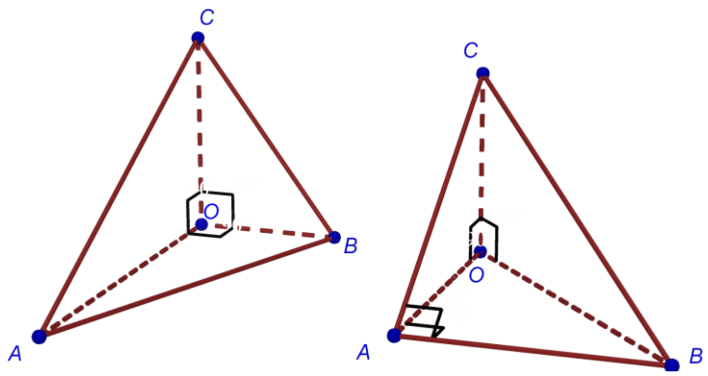

Figure 1: *Tetrahedrons with right planar angles*.

*Corresponding author: dvallo@ukf.sk
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Pythagorean tetrahedron – 1st type

We take up the terminology and derivation of the equation from [2]. We introduce it only on grounds of the understandable explanation for the reader.

We analogously consider the tetrahedron $ABCO$ which edges have labels $OA=a, OB=b, OC=c$, $AB=p$, $BC=q$, $AC=r$ and we also introduce parameters ε,η,ξ of the right-angled triangles ABO, BCO, ACO with planar right angles at vertex O.

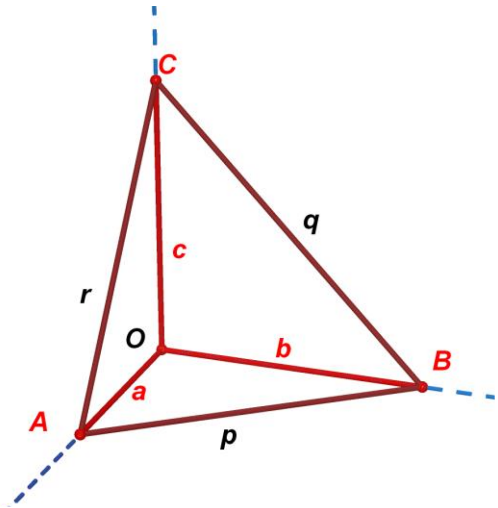

***Figure 2**: Tetrahedron of the first type. It has three right – angled triangles as the faces at vertex O.*

We define the parameters of the edges

$$\varepsilon=\frac{a}{b+p},\ \mu=\frac{b}{c+q} \text{ and } \xi=\frac{c}{a+r}. \qquad (1)$$

It is evident that $0<\varepsilon<1$.
Using the Pythagorean Theorem it is also easy to prove that

a) $1+\varepsilon^2=1+\left(\frac{a}{b+p}\right)^2=\ldots=\frac{2p}{b+p}$,

b) $1-\varepsilon^2=1-\left(\frac{a}{b+p}\right)^2=\ldots=\frac{2b}{b+p}$,

c) $\frac{2\varepsilon}{1-\varepsilon^2}=\ldots=\frac{a}{b}$.

For the other parameters η,ξ hold analogous formulas. This implies that an equation for this type of Pythagorean tetrahedron is

$$\frac{2\varepsilon}{1-\varepsilon^2}\cdot\frac{2\eta}{1-\eta^2}\cdot\frac{2\xi}{1-\xi^2}=\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{c}{a}=1. \qquad (2)$$

The equation (2) represents the test-criterion for computing of the Pythagorean tetrahedron.

If we put $\varepsilon = \frac{c_1}{m_1}$, $\mu = \frac{c_2}{m_2}$ and $\xi = \frac{c_3}{m_3}$,

where $c_i, m_i \in N, i = 1,2,3$, $1 \le c_i \le n$, $2 \le m_i \le n$ for arbitrary positive integer n, then holds true that for $OA = 1$ we can compute rational values of the edges $OB = \frac{1-\varepsilon^2}{2\varepsilon}$, $OC = OB \cdot \frac{1-\eta^2}{2\eta}$.

The reader can find the details and also results in article [2]. The author used the algorithm based on the idea (described above) to compile a computer program. The program ran for $n = 1,2,3,...,25$ and the hardware constructed on vacuum tubes generated 11 different Pythagorean tetrahedrons. From the reason that the calculation has been carried out 29 years ago, we have compiled a similar program and we have obtained the results in Table 1.

***Table 1**: Pythagorean tetrahedron – 1st type for n=1,2,3, ..., 25.*

	a	b	c	p	q	r
1	1100	1155	1008	1595	1533	1492
2	252	240	275	348	365	373
3	720	132	85	732	157	725
4	1584	187	1020	1595	1037	1884
5	240	44	117	244	125	267
6	880	429	2340	979	2379	2500
7	231	792	160	825	808	281
8	480	140	693	500	707	843
9	3536	11220	2925	11764	11595	4589
10	5796	528	6325	5820	6347	8579
11	1008	1100	12075	1492	12125	12117
12	780	2475	2992	2595	3883	3092

If we compare the results with [2], the last result is a new Pythagorean tetrahedron in the list. This is a benefit of this section in our paper.

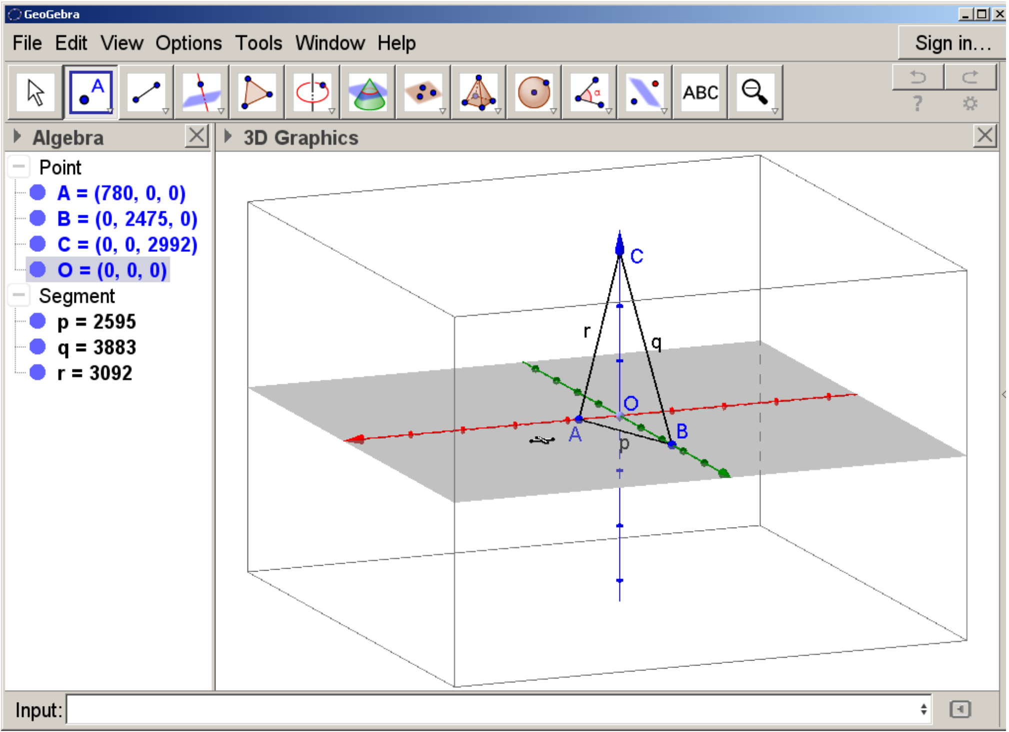

***Figure 3**: The Geogebra sketch of thelast Pythagorean tetrahedron in the list.*

Pythagorean tetrahedron – 2nd type

By analogy to previous case we consider the tetrahedron $ABCO$ which edges have labels $OA = a, OB = b, OC = c$, $AB = p$, $BC = q$, $AC = r$ (see Figure 2).

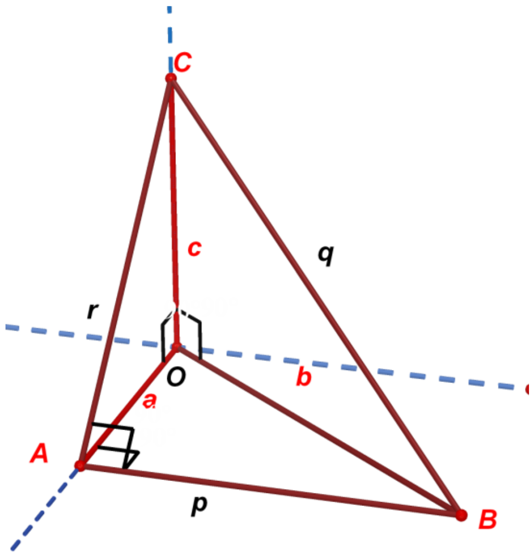

***Figure 4**: Tetrahedron of the second type. It has four right- angles triangles as the faces.*

In this position of the vertices we need not re-define the parameters of the edges and holds (1). Using the Pythagorean Theorem we derive

$$\frac{2\varepsilon}{1+\varepsilon^2}=\frac{a}{b},\ \frac{2\eta}{1-\eta^2}=\frac{b}{c}\ \text{ and }\ \frac{2\xi}{1-\xi^2}=\frac{c}{a}. \qquad (3)$$

This implies that an equation for the 2nd type of Pythagorean tetrahedron is

$$\frac{2\varepsilon}{1+\varepsilon^2}\cdot\frac{2\eta}{1-\eta^2}\cdot\frac{2\xi}{1-\xi^2}=1. \qquad (4)$$

The equation (4) represents the test-criterion for computing of tetrahedron in similar way such as Pythagorean tetrahedron of the 1st type. We have compiled a computer program. The program ran for $n=50$ and we have obtained the results in Table 2.

***Table 2**: Pythagorean tetrahedron – 2nd type for n=1,2,3, ..., 50.*

	ε	μ	ξ	a	b	c	p	q	r
1	**1/13**	**4/5**	**9/17**	104	680	153	672	697	185
2	**1/9**	**6/7**	**13/41**	756	3444	533	3360	3485	925
3	**1/21**	**6/13**	**17/19**	252	2652	2261	2640	3485	2275
4	**1/13**	**9/17**	**4/5**	117	765	520	756	925	533
5	**1/18**	**10/11**	**13/35**	1584	14300	1365	14212	14365	2091
9	**1/21**	**17/19**	**6/13**	399	4199	468	4180	4225	615

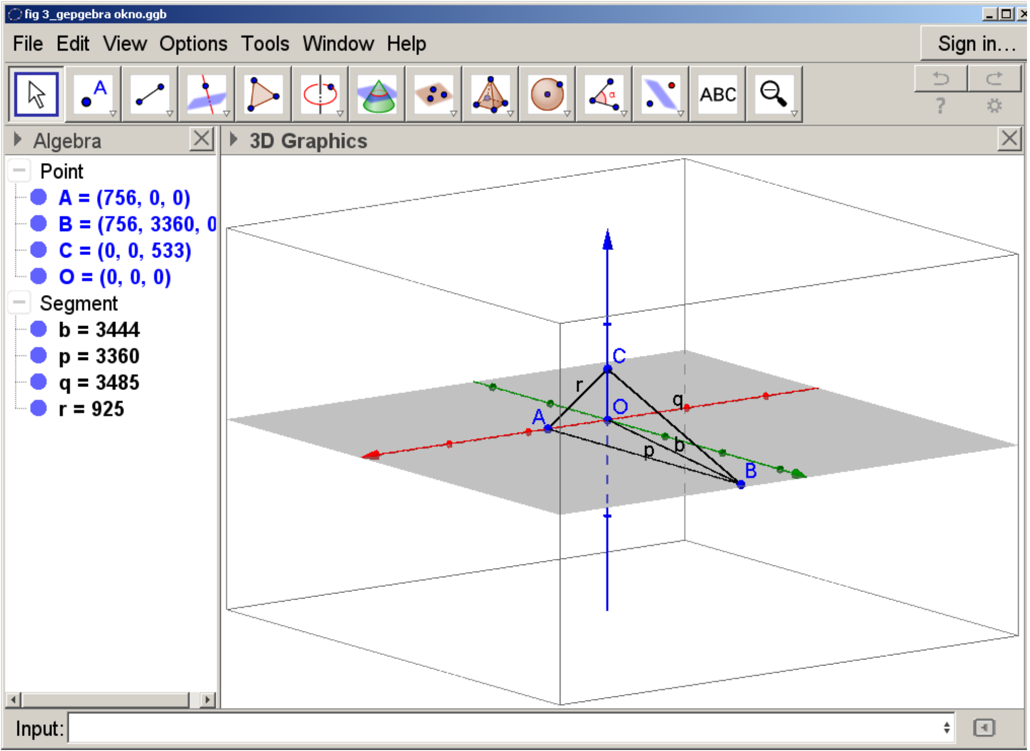

***Figure 5**: The Geogebra sketch of the Pythagorean tetrahedron of the 2nd type.*

Discussion

The idea of this paper comes out from the original paper [2]. Some inquisitiveness leads us to the verification of the results. Using a modern hardware, we have replenished one tetrahedron. The derivation and the computation of the Pythagorean tetrahedron of the 2nd type follows from the analogy.

The reader can find a very interesting problem in [2]. The Pythagorean tetrahedrons of the 1st type can be considered like the grounds of the rectangular parallelepiped in which every triple of edges or planar diagonals form Pythagorean triples, but the solid diagonals have irrational lengths in all cases! Does exist such parallelepiped? This problem is still open.

References

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