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Problem Posing as a Tool for Developing and Designing Tasks

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Abstract

Problem posing is mostly applied during a lesson in such way, that the teacher prepares a material, situation or a problem and the students pose some new problems and solve them. It is possible to use problem posing outside of the classroom. We want to show that problem posing can be successfully used for developing and designing new tasks and problems for mathematics teaching and learning. We describe the What-If-Not strategy of problem posing and give an example of creating new problems using this strategy. We also solve some of the newly created problems.

Keywords: Problem posing, problem solving, task design.

Classification: D50

Introduction

The tasks teachers pose in their classrooms deserve important consideration because they open or close the students' opportunity for meaningful mathematics learning. They convey implicit messages about the nature of mathematics: what it is, what it entails, and what is worth knowing and doing in mathematics (Crespo, 2003). This is a responsibility not only of the teachers but also authors of textbooks and task designers, because teachers pose problems that come from textbooks or other literature. Now a question arises, where do the problems in a textbook come from? In most cases, they are a product of a creative thinking of the authors or anyone, who prepares a new problems for teaching and learning. We assume that adopting principles and strategies of problem posing can help them in generating original problems.

Problem-posing

Problem posing involves generating of new problems and questions aimed at exploring a given situation as well as the reformulation of a problem during the process of solving it (Silver, 1994). In the first case, problem posing is a divergent task that has multiple possible answers (posed problems). Therefore, problem posing is considered to be a creative generation task that requires productive thinking. To promote diverse and flexible thinking, it is critical for learners to generate diverse problems. However, it has been confirmed that problems generated by novice learners lack diversity (Kojima & Miwa, 2008). Advocates for problem posing typically argue that experience with mathematical problem posing can promote students' engagement in authentic mathematical activity; allow them to encounter

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many problems, methods and solutions, and promote students' creativity – a disposition to look for new problems, alternate methods, and novel solutions (Singer et al, 2011).

A systematic training focused on problem transposition using various representations, problem extension by adding new operations or conditions, comparison of various problems in order to assess similarities and differences, or analysis of incomplete or redundant problems can raise students' awareness of meaningful problems (Singer et al, 2011). Effective teaching should focus on representational change, within a variety of activities, which specifically address students' motor, visual, and verbal skills, as well as transfers in between them (Singer et al, 2011). In problem-posing contexts, students are stimulated to make observations, experiment through varying some data and analyzing the results, and devise their own new problems that could be solved by equally using similar or different patterns. The processes by which students continue given series or patterns provide information about the cognitive approaches they use in problem solving (Singer, Voica, 2008)

If problem posing is such an important intellectual activity, the first question we need to ask is who can pose mathematical problems. One important line of research in problem posing has been exploring what problems teachers and students can pose (Singer et al, 2011). The process of posing problems in this line proceeds during a mathematics lesson, where a problem situation is given and students are asked to pose new problems. The process of problem posing can be also used out of the lesson by teachers, authors of textbooks or task designers.

What-If-Not strategy

The What-If-Not (WIN) strategy was introduced by Brown and Walter (2005). It is based on the idea that modifying the attributes of a given problem could yield new and intriguing problems which eventually may result in some interesting investigations. The strategy cen be described in following stages:

Level 0 – Choosing a Starting Point: As a starting point for problem posing can be used a concrete material, existing problem or a theorem.

Level 1 – Listing Attributes: Look at what is given and find the important attributes of the starting point.

Level 2 – What-If-Not-ing: Choose one or more attributes from the list and ask "What if not this attribute?" and list some alternatives for this attribute.

Level 3 – Question Asking or Problem Posing: Choose an alternative to the attribute and pose a question or problem.

Level 4 – Analyzing the Problem: Analyzing and trying to answer the question gives us a deeper insight into the problem.

The process of the WIN strategy seems to be linear, but it is cyclical in fact. Brown and Walter (2005) explain: "Our scheme, however, is not as linear as it may seem from this list. Almost every part can (and does) affect others. A new question may trigger a new attribute, and a new attribute may in turn trigger a new question (for example). This in turn may enable you to see the original phenomenon in a new light."

Using problem-posing for task design

Many projects or final thesis at universities have a similar objective, to create a collection of new task for a certain part of mathematics or a specific type of tasks. Strategies of problem-posing can be used to fulfil this objective. Authors often unwittingly and intuitively use the WIN strategy to create new tasks. We suggest that intentional usage of problem-posing strategies can increase quantity and quality of designed tasks.

Now we show how to develop new problems using the WIN strategy. As a starting point we choose a problem we found in (Kopka, 2010). In the following problems, we call a square with a side of length k a square of size $k \times k$ and similarly a rectangle of size $k \times l$ is a rectangle with sides of length *k* and *l*. An equilateral triangle with the side of length *k* is called a triangle of size *k*.

<u>*Problem*</u>: Determine the number of all squares in a square grid of size $n \times n$, where *n* is a natural number other then 0. (Kopka, 2010, p.124)

Now we need to create a list of attributes of this problem. In other word we can describe the situation that we have a <u>square</u> with side of <u>length *n*</u> divided into <u>squares</u> with side of <u>length 1</u> and we want to know the number of <u>all</u> <u>squares</u> in the grid. In the previous sentence we underlined the most important attributes of the problem. Now we can start what-if-not-ing. We write down some possible alternatives for each attribute:

Square of size $n \times n$ – triangle of size *n*, rectangle of size $m \times n$, square of size $n \times n$ with a missing square in a corner(s).

Squares of size 1×1- triangles of size 1, squares of different sizes and their combination.

Number of all squares – number of squares of certain size, number of all triangles, number of all rectangles.

Choosing some of the alternatives we can pose the following problems:

<u>Problem 1</u>: Count the number of all triangles in a triangular grid of size *n*.

<u>Problem 2</u>: Count the number of all rectangles in a square grid of size $m \times n$.

<u>Problem 3</u>: Count the number of all squares in a square grid of size $n \times n$ divided into one square of size 2×2 and squares of size 1×1 (see Figure 1, left).

<u>Problem 4</u>: Count the number of all squares in a square grid of size $n \times n$ divided into one square of size 2×2 and squares of size 1×1, where the square of size 2×2 is placed arbitrarily in the grid..

<u>Problem 5.</u> Count the number of all squares in a square grid of size $n \times n$ with one square missing in the up-left corner (see Figure 1, right).

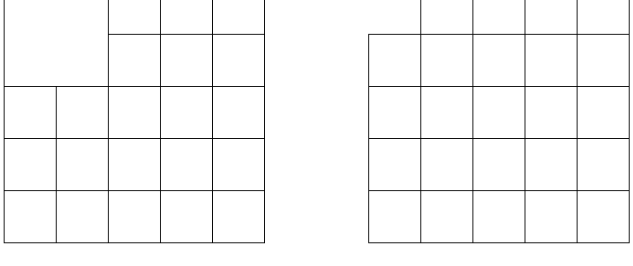

Figure 1

It is possible to pose even more new problems by varying the attributes of the original problem. Of course replacing one attribute may lead to a trivial problem so it needs to also replace another attribute too. For example if we replace the number of all squares by triangles, the new problem has no solution.

We assume that the stated problems fully present the potential of WIN strategy for generating new problems. As a last step of the WIN strategy, we analyse and solve four of these problems. In the solutions, we use the following well known identities for finite sums.

$$\sum_{k=1}^{n} k = \frac{1}{2}n(n+1), \sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1), \sum_{k=1}^{n} k(k+1) = \frac{1}{3}n(n+1)(n+2).$$

<u>Problem 1:</u> Count the number of all triangles in a triangular grid of size *n*.

Solution: In such a triangle there are triangles of different sizes and some are rotated about 180 degrees. We will try to count all these types of triangles separately. First we focus on the triangles in the basic position. We start with triangles of size 1. We can notice that in the first row there is only 1 such triangle. In the second row there are 2. In the i-th row there are *I* triangles of size 1. So the number of triangles of size 1 is $1+2+3+\cdots+n$, which we denote by T_n, the triangular number. Now we will count the triangles of size 2, by moving the triangle in all the possible positions. We will track the position of the upper triangle of size 1 in the triangle of size 2 (see Figure 2). The tracked triangle occupies the positions of a triangle of size 1 in the first $n-1$ rows. Therefore the number of triangles of size 2 is T_{n-1}. This idea can be generalized for any triangle of size *I* giving the result T_{n-i+1}.

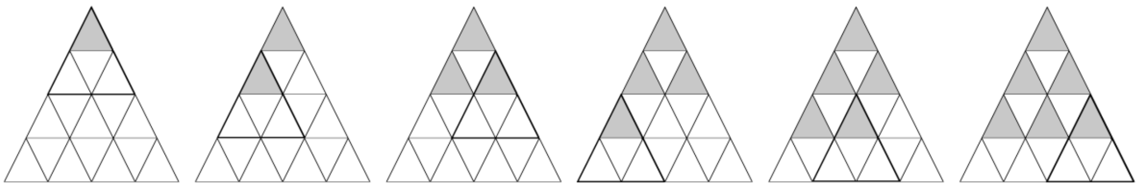

Figure 2

Now we will focus on the rotated triangles. First of all we need to realize that not all sizes are possible for the rotated triangles. The side of a rotated triangle cannot be longer then $\frac{n}{2}$. If *n* is odd the largest size is equal to $\frac{n-1}{2}$, for even *n* it is $\frac{n}{2}$. We can write it jointly $\left\lfloor\frac{n}{2}\right\rfloor$. Now we can count the triangles similarly as in the case of triangles in the basic position. The number of triangles of size 1 is in row *I* equal to $i-1$. So summing it up gives T_{n-1}. For triangles of size 2 we will track the position of the triangle of size 1 in the basic position in the middle. It is obvious that the tracked triangle will not occupy any position in the first two rows and in the last row. All the possible positions of the tracked triangle are the triangles of size 1 in the basic position in a triangle of size $n-3$. The number of rotated triangles of size 2 is equal T_{n-3}. We can generalize this for triangles of size *i*, where *I* is a natural number satisfying inequality $1 \le i \le \left\lfloor\frac{n}{2}\right\rfloor$.

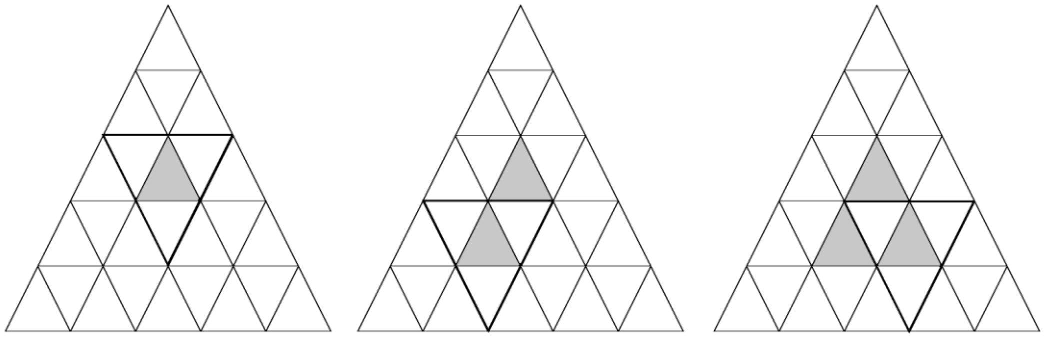

Figure 3

We summarize our findings in Table 1.

Table 1

Size		1	2	3	4	…	$n-1$	n
Position	Basic	T_n	T_{n-1}	T_{n-2}	T_{n-3}	…	T_2	T_1
	Rotated	T_{n-1}	T_{n-3}	T_{n-5}	T_{n-7}	…	0	0

We want to derive an explicit formula for counting the number of triangles. First we count the number of triangles in the basic position. We know that $T_n = \frac{1}{2}n(n+1)$. Then

$$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n} \frac{k(k+1)}{2} = \frac{1}{2}\sum_{k=1}^{n} k(k+1) = \frac{1}{6}n(n+1)(n+2).$$

We need to distinguish two cases for counting the rotated triangles. If *n* is even, we sum up triangular numbers with odd index.

$$\sum_{k=1}^{n} T_{2k-1} = \sum_{k=1}^{n} \frac{2k(2k-1)}{2} = \sum_{k=1}^{n} k(2k-1) = 2\sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} k =$$

$$= \frac{1}{3}n(n+1)(2n+1) - \frac{1}{2}n(n+1) = \frac{1}{6}n(n+1)(4n-1).$$

If *n* is odd, we sum up triangular numbers with even index.

$$\sum_{k=1}^{n} T_{2k} = \sum_{k=1}^{n} \frac{2k(2k+1)}{2} = \sum_{k=1}^{n} k(2k+1) = 2\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k =$$

$$= \frac{1}{3}n(n+1)(2n+1) + \frac{1}{2}n(n+1) = \frac{1}{6}n(n+1)(4n+5).$$

We denote by $t(n)$ the number of all triangles in the triangular grid of size *n*. For *n* even we have:

$$t(n)=\sum_{k=1}^{n}T_k+\sum_{k=1}^{n/2}T_{2k-1}=\frac{1}{6}n(n+1)(n+2)+\frac{1}{6}\frac{n}{2}\left(\frac{n}{2}+1\right)\left(4\frac{n}{2}-1\right)=$$
$$=\frac{1}{8}n(n+2)(2n+1).$$

For *n* odd we have:

$$t(n)=\sum_{k=1}^{n}T_k+\sum_{k=1}^{(n-1)/2}T_{2k}=\frac{1}{6}n(n+1)(n+2)+\frac{1}{6}\frac{n-1}{2}\left(\frac{n-1}{2}+1\right)\left(4\frac{n-1}{2}+5\right)=$$
$$=\frac{1}{8}(n+1)(2n^2+3n-1).$$

For example we can count $t(4)$ and $t(5)$.

$$t(4)=\frac{1}{8}4.6.9=27,\qquad t(5)=\frac{1}{8}6.64=48.$$

Problem 2: Count the number of all rectangles in a square grid of size $m\times n$.

Solution: We can find rectangles of different sizes in the rectangle. Some of them are squares. We will distinguish between rectangles of size $a\times b$ and $b\times a$. If we have a rectangle of size $a\times b$, where $1\le a\le m$ and $1\le b\le n$, we can place it in the rectangle in such way that their top left corners merge into one. This is one possible way of placing the rectangle of $a\times b$. Now we can move it to the right side into one of the $m-a$ positions. This gives us $m-a+1$ positions in the horizontal line. Similarly we can move the rectangle in the vertical line. There we obtain $n-b+1$ positions. The number of positions of the rectangle of size $a\times b$ in a rectangle of size $m\times n$ is $(m-a+1)(n-b+1)$. Now we put together a table in which each cell contains the number of rectangles of size $a\times b$ (see Table 2). The last column contains sums of rows.

Table 2

Size	1	2	…	$m-1$	m	Sum
1	mn	$(m-1)n$	…	$2n$	n	$n(1+2+\cdots+m)$
2	$m(n-1)$	$(m-1)(n-1)$	…	$2(n-1)$	$n-1$	$(n-1)(1+2+\cdots+m)$
⋮	⋮	⋮		⋮	⋮	⋮
$n-1$	$2m$	$2(m-1)$	…	4	2	$2(1+2+\cdots+m)$
n	m	$(m-1)$	…	2	1	$(1+2+\cdots+m)$

We denote by $r(m,n)$ the number of all rectangles. We need to sum up all the expressions in the last column of Table 2.

$$r(m,n)=m(1+2+\cdots+n)+(m-1)(1+2+\cdots+n)+\cdots+(1+2+\cdots+n)=$$
$$=(1+2+\cdots+m)(1+2+\cdots+n).$$

We can also derive a recurrence for $r(m,n)$. If we take a look on Table 3, we can see that it contains same expressions as Table 2 except the first row and first column.

We can write:

$$r(m+1,n+1) = r(m,n) + (n+1)\big(1+2+\ldots+(m+1)\big) + (m+1)(1+2+\cdots+n) =$$
$$= r(m,n) + \frac{1}{2}(n+1)(m+1)(m+2) + \frac{1}{2}(m+1)n(n+1) =$$
$$= r(m,n) + \frac{1}{2}(m+1)(n+1)(m+n+2).$$

We need to determine $r(1,n)$ and $r(m,1)$. If we look into Table 2 from column *m* and row *n* we obtain:

$$r(m,1) = 1+2+\cdots+m = \frac{1}{2}m(m+1).$$
$$r(1,n) = 1+2+\cdots+n = \frac{1}{2}n(n+1).$$

Table 3

	1	2	3	…	m	$m+1$
1	$(m+1)(n+1)$	$m(n+1)$	$(m-1)(n+1)$		$2(n+1)$	$(n+1)$
2	$(m+1)n$	mn	$(m-1)n$	…	$2n$	n
3	$(m+1)(n-1)$	$m(n-1)$	$(m-1)(n-1)$	…	$2(n-1)$	$n-1$
⋮	⋮	⋮	⋮		⋮	⋮
n	$2(m+1)$	$2m$	$2(m-1)$	…	4	2
$n+1$	$(m+1)$	m	$(m-1)$	…	2	1

For example we can count $r(5,3)$. By the explicit formula we have:

$$r(5,3) = (1+2+3+4+5)(1+2+3) = 15.6 = 90.$$

Using the recurrence we obtain:

$$r(5,3) = r(4,2) + \frac{1}{2}5.3.8 = r(3,1) + \frac{1}{2}4.2.6 + 60 = 6 + 24 + 60 = 90.$$

Remark: If we choose $m = n$, we have a similar problem of determining the number of rectangles in a square of size $n \times n$. Solution of this problem is the same, but the derived formulas have the form:

$$r(n) = (1+2+\cdots+n)^2 = \sum_{i=1}^{n} i^3 = \frac{1}{4}n^2(n+1)^2$$
$$r(n+1) = r(n) + (n+1)^3, r(1) = 1.$$

In problems 3 and 5 we use the result of the original problem of counting the number of all squares in a square grid. We denote it by $s(n)$. It can be derived even from table 2 looking at the cells laying at the main diagonal and setting $m = n$. We see that $s(n) = \sum_{k=1}^{n} k^2$.

<u>Problem 3</u>: Count the number of all squares in a square grid of size $n \times n$ divided into one square of size 2×2 and squares of size 1×1 (see Figure 1, left).

Solution: We will count, how many squares are missing in this kind of grid. There are missing 4 squares of size 1×1. The square of size $n \times n$ is still in the grid. As for the squares of other sizes, there are always 3 missing squares of each size. Together it is $3n - 2$ missing squares. If we denote $s_1(n)$ the number of all squares in this kind of a square grid, then:

$$s_1(n) = s(n) - (3n - 2) = \frac{1}{6}n(n+1)(2n+1) - (3n-2) = \frac{1}{6}(n-1)(n+4)(2n-3).$$

Problem 5. Count the number of all squares in a square grid of size $n \times n$ with one square missing in the up-left corner (see Figure 1, right).

Solution: We will count how many squares disappear by removing the square from the corner. All squares in a square grid that are placed in such way that they cover the removed square, do not appear in the grid any more. Number of these squares is exactly *n*, because there is only one square of each size that can be placed in a square grid in the described way. If we denote $s_2(n)$ the number of squares in the square grid without a square in the corner, then:

$$s_2(n) = s(n) - n = \frac{1}{6}n(n+1)(2n+1) - n = \frac{1}{6}n(n-1)(2n+5).$$

Conclusion

We think that problem posing and especially the WIN strategy can be a very useful for developing new original tasks and problem for mathematics teaching and learning. We showed an example of using the WIN strategy and formulated five new problems of which we solved four. We could pose more problems, but the stated problems fulfil our objective to point out a possible use of problem posing.

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