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Some Aspects of the Relation between History of Mathematics and Mathematics Education, the Case of Infinite Series

Jozef Fulier*

Department of Mathematics, Faculty of Natural Sciences, Constantine the Philosopher University in Nitra, Tr. A. Hlinku 1, SK-949 74 Nitra,

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Abstract

The paper deals with some aspects of the relationship between history of mathematics and mathematics education. Attention is paid to the importance of integrating elements of history of mathematics in preparation of prospective mathematics teachers. The emphasis is put on personal experience from teaching the university course history of mathematics for prospective teachers and implementation of elements of mathematics history in calculus courses for prospective mathematics teachers. We primarily focus on infinite series because this concept is mysterious and intriguing.

Keywords: History of mathematics, mathematics education, infinite series.

Classification: A30, I30

Introduction

Mathematics in relation to historical periods can be viewed in two ways. On one hand we see it as a system of interconnected timeless eternal facts. In this respect mathematics does not know “outdated” knowledge, in contrast to other natural sciences. Mathematical argument which was once correctly proved (since then it is referred to as a mathematical theorem) never loses this value, although it can happen (and it also occurs in general) that the further development becomes a simple case of more general claim. It cannot, therefore, be surprising that the average person hardly thinks that mathematics has any history, rather, that all of it was revealed in a flash of moment to some *ancient mathematical Muses*. In this sense, it is important to show students that mathematics exists and evolves in time and space. We would like to show that it is a science that has undergone an evolution rather than something which arose out of thin air, and stress that human beings have taken part in this evolution and that the evolution of mathematics has been influenced by many different cultures throughout history and that these cultures have had an influence on the shaping of mathematics as well as the other way round. Mathematicians’ activity is focused on the discovery of these timeless facts and clarification of the deductive connection between them. In essence it is about evolutionary progress during which mathematics is becoming better and better. On the other hand, we can look at mathematics as a human activity taking place in the cultural time. This second view is not inconsistent with the first view, quite the opposite,

*Corresponding author: jfulier@ukf.sk
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relevant knowledge effectively complements it by clarifying the logic of discoveries. This allows us to see that the activity with which mathematicians discover new knowledge has much in common with activity that teachers and students do when dealing with the revealed knowledge. Indeed, even in this activity in which teachers are trying to make sense of the learning process, there are elements of creativity and real discoveries. The traditional form of teaching mathematics does not use historical approach in learning. Thus, it is not surprising that at Slovak universities a one-semester course of history of mathematics for prospective mathematics teachers is included in the last year of their master study. The mathematics history lectures in the last year of prospective teacher preparation assumed that history of mathematics cannot enrich the comprehensive knowledge or skill in the subject of mathematics. However, the current mathematics is too abstract and completely separated from its historical roots that may support its understanding. This abstractness of mathematics is also transferred in mathematics education and makes mathematics learning even more difficult. Perhaps a way out of this situation can be to change the view on the relationship of mathematics and its history, and to start systematically investigate the origin and genesis of mathematical ideas, not only in the final course of the history of mathematics, but directly in key university mathematics courses (calculus, algebra, geometry).

The role of mathematics history in mathematics teaching and learning

In general mathematics is viewed as a collection of methods and problems. In our opinion reducing mathematics to this aspect is a distorted picture. Mathematics is much more: it is part of our culture, just as literature, music, philosophy, arts. The cultural aspect of these subjects has been underlined in school by teaching also their historical development. Over the years mathematicians, educators and historians have wondered whether mathematics learning and teaching might profit from integrating elements of history of mathematics. It is clear that mathematics education does not succeed to reach its aims for all students, and that it is therefore worthwhile to investigate whether history can help to improve the situation. The idea to use the history of mathematics in mathematics education is not new. The idea has already been explicitly described in the works of *Heppel*, 1893; *Smith* and *Cajori*, 1894; *Loria* 1899; *Zeuthen,* 1902; *Gebhardt*, 1912. In 1902, Dutch mathematician *H. G. Zeuthen* (1839-1920) published the *History of Mathematics* for teachers. *Zeuthen* argued that the history of mathematics should be part of general teachers' education. Since then, math teachers increasingly use the history of mathematics in their curricula, and the spectrum of its use has spread. Since 1960, research in this area has founded the scientific basis. What is now called "**HPM**" sprang from a Working Group established at the second *International Congress on Mathematical Education* (ICME), held in Exeter, UK, in 1972. The "principal aims" of the Study Group were proposed as follows:

1. To promote international contacts and exchange information concerning: a) Courses in History of Mathematics in Universities, Colleges and Schools. b) The use and relevance of History of Mathematics in mathematics teaching. c) Views on the relation between History of Mathematics and Mathematical Education at all levels. 2. To promote and stimulate interdisciplinary investigation by bringing together all those interested, particularly mathematicians, historians of mathematics, teachers, social scientists and other users of mathematics. 3. To further and deeper understanding of the way mathematics evolves, and the forces which contribute to this evolution. 4. To relate the teaching of mathematics and the history of mathematics teaching to the development of mathematics in ways which assist the improvement of instruction and the development of curricula. 5. To

produce materials which can be used by teachers of mathematics to provide perspectives and to further the critical discussion of the teaching of mathematics. 6. To facilitate access to materials in the history of mathematics and related areas. 7. To promote awareness of the relevance of the history of mathematics for mathematics teaching in mathematicians and teachers. 8. To promote awareness of the history of mathematics as a significant part of the development of cultures.

Educators throughout the world have been formulating and conducting research on the use of history of mathematics in mathematics education. In the last three or four decades there has been a movement towards inclusion of more humanistic elements in the teaching of mathematics. This has been the case of Slovakia, in particular the Slovak upper secondary schools, as well as internationally. The various 'humanistic' elements embrace, among others, cultural, sociological, philosophical, application-oriented, and historical perspectives on mathematics as an educational discipline*. Only recently there has been a stronger call for methodological and theoretical foundations for the role of history in mathematics education. The report *History in Mathematics Education: The ICMI† Study* (*Fauvel & Van Maanen*, 2000) has made a valuable contribution in this respect by collecting theories, results, experiences and ideas of implementing history in mathematics education from around the world. Students can experience the subject as a human activity, discovered, invented, changed and extended under the influence of people over time. Instead of seeing mathematics as a ready-made product, they can see that mathematics is a continuously changing and growing body of knowledge to which they can contribute themselves. Learners could acquire a notion of processes and progress and learn about social and cultural influences. Moreover, history accentuates the links between mathematical topics and the role of mathematics in other disciplines, which would help place mathematics in a broader perspective and thus deepen students' understanding. History of mathematics may play an especially important role in the training of future teachers, and also teachers undergoing in-service training. There are several reasons for including a historical component in such training, including the promotion of enthusiasm for mathematics, enabling trainees to see pupils differently, to see mathematics differently, and to develop skills of reading, library use and expository writing which can be neglected in mathematics courses. It may be useful here to distinguish the training needs for primary, secondary and higher levels. (ICMI Study on The role of the history of mathematics in the teaching and learning of mathematics, Discussion Document). History of mathematics provides opportunities for getting a better view of what mathematics is. When a teacher's own perception and understanding of mathematics changes, it affects the way mathematics is taught and consequently the way students perceive it. Teachers may find that information on the development of a mathematical topic makes it easier to explain or give an example to students. *Jankvist* (2009) in his PhD thesis states that in general the arguments for using history are of two different kinds: those that refer to history as a tool for assisting the learning and teaching of mathematics, and those that refer to history as a goal. Each of these two kinds constitutes its own category of arguments. The category of history-as-a-tool arguments

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*Note that in the case of Slovakia, this is a little bitter undertone, as efforts to humanize the teaching of mathematics, together with efforts to improve the performance of pupils Slovak testing OECD PISA in 2008 resulted in the reform of educational curriculum, which, though committed to the above mentioned objectives, significantly reduced the number of lessons in individual grades of mathematics education, thus, the initial idea was lost.

†*ICMI = International Commission on Mathematical Instruction*

contains the arguments concerning how students learn mathematics. A typical argument is that history can be a motivating factor for students in their learning and study of mathematics, for instance, helping to sustain the students' interest and excitement in the subject. Or, that an historical approach may give mathematics a more human face and therefore make it less frightening. Often pieces of the mathematical development over which past mathematicians have stumbled are also troublesome for nowadays students of mathematics. Students may derive comfort from knowing that the same mathematical concept which they themselves are now having trouble grasping actually took great mathematicians hundreds of years to shape into its final form. Besides having these motivational and more affective effects, history may also play the role of a cognitive tool in supporting the mathematics learning itself. For instance, one argument states that history can improve learning and teaching by providing a different point of view or mode of presentation. Other arguments say that historical phenomenology may prepare the development of a hypothetical learning trajectory, or that history "can help us look through the eyes of the students". In the context of using history to study learning processes we mention the so-called *Biogenetic Law* popular at the beginning of the 20th century. German biologist and natural philosopher *E. Haeckel* in 1874 formulated his theory as "*Ontogeny recapitulates phylogeny*". The notion later became simply known as *the recapitulation theory*. Ontogeny is the growth (size change) and development (shape change) of an individual organism; phylogeny is the evolutionary history of a species. *Haeckel* claimed that the development of advanced species passes through stages represented by adult organisms of more primitive species. In other words, each successive stage in the development of an individual represents one of the adult forms that appeared in its evolutionary history. Although Haeckel's specific form of recapitulation theory is now discredited among biologists, the strong influence it had on social and educational theories of the late 19th century still resonates in the 21st century. Haeckel developed this thought even further saying that "the psychic development of a child is a brief repetition of the phylogenetic evolution". And it is this argument that translates into the recapitulation argument which may be formulated as: To really learn and master mathematics, one's mind must go through the same stages that mathematics has gone through during its evolution. The Biogenetic Law states that mathematical learning in the individual (*ontogenesis*) follows the same course as the historical development of mathematics itself (*phylogenesis*). Developmental psychologist *Jean Piaget* (1896 – 1980) favoured a softer version of the formula, according to which ontogeny parallels phylogeny because the two are subject to similar external constraints. However, it has become more and more clear since then that such a strong statement cannot be sustained.

The plenary lecture given to the Congress (ICME 4, Berkeley 1980) by the distinguished Dutch mathematics educator *Hans Freudenthal* (1905 – 1990) valuably included his succinct views on the "*ontogeny recapitulates phylogeny*" debate which has long been a concern to those in HPM circles:

Figure 1*:* Jean Piaget (1896 – 1980)

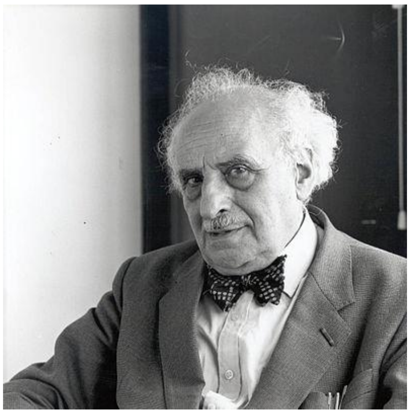


Figure 2*:* Hans Freudenthal (1905 -1990)

"History of mathematics has been a learning process of progressive schematizing. Youngsters need not repeat the history of mankind but they should not be expected either to start at the very point where the preceding generation stopped. In a sense youngsters should repeat history though not the one that actually took place but the one that would have taken place if our ancestors had known what we are fortunate enough to know."(Freudenthal, 1980). A short study of mathematical history is sufficient to conclude that its development is not as consistent as this law would require. *Freudenthal* explains what he understands by "guided reinvention"*: "Urging that ideas are taught genetically does not mean that they should be presented in the order in which they arose, not even with all the deadlocks closed and all the detours cut out. What the blind invented and discovered, the sighted afterwards can tell how it should have been discovered if there had been teachers who had known what we know now. (...) It is not the historical footprints of the inventor we should follow but an improved and better guided course of history."*(*Freudenthal,* 1973).
The recapitulation argument not only applies to mathematics as a whole, but also to single mathematical concepts and theories. And it is often in relation to the development of single mathematical concepts that another tool argument related to the evolutionary kind, the so-called historical parallelism, is put to the "test" – historical parallelism concerns the observation of difficulties and obstacles that occurred in history reappearing in the classroom. The idea of parallelism may also be used as a methodology or heuristic to generate hypotheses in mathematics education (e.g. *Fauvel &van Maanen*, 2000, p. 160;). The German mathematician *Otto Toeplitz*[*] (1881-1940) proposed and distinguished between the "*Direct Genetic Method*" and the "*Indirect Genetic Method*". The "*Indirect Genetic Method*" means that the teacher can learn from the history about difficulties which were encountered even by the great mathematicians such as *Newton, Leibniz, Fermat, Cavalieri* and others and to take this into account in his planning of the teaching process without mentioning historical details.

[*] In 1949 (in German) and 1963 (in English) his textbook *The Calculus A genetic approach* was published posthumously. This book presented a radically different approach to the teaching of calculus. In sharp contrast to the methods of his time, *Otto Toeplitz* did not teach calculus as a static system of techniques and facts to be memorized. Instead, he drew on his knowledge of the history of mathematics and presented calculus as an organic evolution of ideas beginning with the discoveries of Greek scholars, such as *Archimedes, Pythagoras*, and *Euclid*, and developing through the centuries in the work of *Kepler, Galileo, Fermat, Newton,* and *Leibniz*. Through this unique approach, *Toeplitz* summarized and elucidated the major mathematical advances that contributed to modern calculus.

The historical development only acts as a guideline. It shows the teacher (or the textbook author) the crucial way forward: namely, that those aspects of a concept which historically have been recognised and used before others are probably more appropriate for the beginning of teaching than modern deductive reformulations. The genetic method that is going back to the roots of the concepts can offer a way beyond the dilemma of rigour versus intuition in teaching. In contrary to this the "*Direct Genetic Method*" proposes in addition to offer historical details as well (often only a few sentences or a single historical problem) explicitly in the teaching at suitable occasions (*Kronfellner,* 2000). *Schubring* (1978), in his extensive study hereof, distinguishes between two genetic principles: (1) the historical-genetic principle, which aims at leading students from basic to complex knowledge in the same way that mankind has progressed in the history of mathematics, and (2) the psychological-genetic principle, which is based on the idea to let the students rediscover or reinvent mathematics by using their own talent and experiences from the surrounding environment. (*Jankvist*, 2009).

Calculus, infinite series

Calculus fine example of mathematical disciplines in teaching that apply to supplements of *Freudenthal* to Biogenetic Law (mathematical learning in the individual (ontogenesis) follows the same course as the historical development of mathematics itself (phylogenesis) in such a large extent that we can say the teaching of mathematical analysis is currently being implemented under the "*Anti Biogenetic Law*". Indeed, according *Hairer& Wanner* (2008):

"*Traditionally, a rigorous first course in Analysis progresses (more or less) in the following order:* ***sets, mappings ⇒ limits, continuous functions ⇒ derivatives ⇒ integration****. On the other hand, the historical development of these subjects occurred in reverse order:* ***Cantor 1875, Dedekind ⇐ Cauchy 1821, Weierstrass⇐ Newton 1665, Leibniz 1675⇐ Archimedes, Kepler 1615, Fermat 1638."***

University calculus course, especially function and limits of function, cause serious problems to students. On the other hand, it is known from history that terms function and limits of function using *ε-δ notation* were introduced to mathematics in the end of 19th century by German mathematicians *P. G. L. Dirichlet* (1805 -1859) and *K.* Weierstrass (1815 - 1897). The start of using this notation meant the final step in such a very important period that lasted for several centuries. Within this development intuitive and easily understandable terms were substituted by less visual and understandable terms. That is why, as presented by L. Kvasz, lot of students can not translate into their own language everything that they hear in the lectures. The secondary schools mathematics ends up at the level of the 17th century mathematics (with polynomials and systems of equations), but university courses start with the 19th century mathematics notation and language. It means that two hundred years were cut out of syllabus. We also need to mention that this period meant a kind of stagnation for algebra, but for calculus it was a time of rapid development. During this time several approaches to terms like function, limit of function, derivative and integral were used.

For students it means that this approach is hardly understandable because they do not understand the reason why the general term of function and limits of function have been defined by ε-δ notation during the lectures. This gap in understanding can be nicely bridged by history of mathematics, to study the historical development of the terms or theory. Based on this situation we can see the history of mathematics as a bridge that helps prospective teachers to overcame the gap that is between the mathematical concepts developed during secondary education and university courses.

Infinite series

The theory of infinite series is an especially interesting mathematical construct due to its wealth of surprising results. In its most basic setting, infinite series is vehicle we use to extend the finite addition to the "infinite addition". The standard presentation of infinite series in calculus courses in Slovakia as taught today is as follows:

(1) A short introduction to infinite sequences and their limits, convergent and divergent sequences;
(2) Abstract definitions of infinite series $\sum_{n=1}^{\infty} a_n$, with convergence defined in terms of limits of sequences of partial sums $\lim\limits_{n\to\infty} s_n =: s$, where $s_n = a_1 + a_2 + \cdots + a_n, n \in \mathbb{N}$. If $s \in \mathbb{R}$, then we say that series $\sum_{n=1}^{\infty} a_n$ converges to the sum s, and we write $\sum_{n=1}^{\infty} a_n = s$;
(3) Theorems and convergence tests for positive term series; for alternating series and for general series;
(4) Definition and theorems about power series and general functional series.

The emphasis is put on convergence and especially on convergence tests. We spend our time investigating whether series converge or not, but little or no time investigating what the series converge to. The preoccupation with determining convergence but not the sum makes the whole process seem artificial and pointless for many students, and instructors as well. (*Lehmann*, 2000).

At *12th Nitra conference* a lecturer wrote in her lectures on computer graphics some equalities in the form of divergent series (for example $1 + 2 + 3 + \cdots + n + \cdots = -\frac{1}{12}$) derived by *Leonhard Euler* (1707 – 1783). Most of the participating students seemed amused, even shaking their heads in disapproval. The noise which followed and the expression in their faces said: "The equality cannot hold, because these series is divergent. It does not make sense. How is it possible that Euler, one of the greatest mathematicians in history, did not know this? But the first year students already know this!"

This situation can be commented by *D. J. Struik* (1948) that: "we cannot always follow Euler when he writes that $1 - 3 + 5 - 7 + \cdots = 0$, or when he concludes from $n + n^2 + \cdots = \frac{n}{1-n}$, and $1 + \frac{1}{n} + \frac{1}{n^2} + \cdots = \frac{n}{n-1}$ that $\ldots + \frac{1}{n^2} + \frac{1}{n} + 1 + n + n^2 + \cdots = 0$. In this situation we must be careful and not too much criticize Euler for his way of manipulating divergent series; he simply did not always use some of our present tests of convergence or divergence as a criterion for the validity of his series. It is known that despite the incredible amount of work that Euler did, he also wrote occasionally some things that were wrong. Euler's foundation of the calculus may have had some weakness, but he expressed his point of view without vagueness. A lot of his excellent work with series has been given a high credit by modern mathematicians." A partial answer to this question can be found in the history of infinite series. In the first place, it is good to remark that famous definitions of infinite series $\sum_{k=1}^{\infty} a_n$, with convergence defined in terms of limits of sequences of partial sums, was formulated by the excellent French mathematician *Augustin-Louis Cauchy* (1789 - 1857) in his textbook *Cours d'Analyse* in 1821. Almost all of modern definitions of convergence of infinite series copy *Cauchy's* words formulated in his *Cours d'Analyse*:

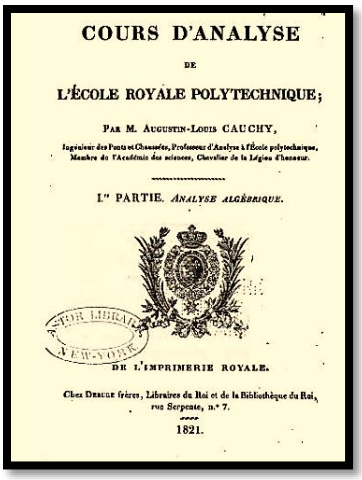


***Figure 3**:* Cours d'Analyse (1821)

***Figure 4**:* Augustin Louis Cauchy (1789 – 1857)

"We call a series an indefinite of quantities, $u_0, u_1, u_2, u_3, \ldots$, which follow from one to another according to a determined law. These quantities themselves are the various terms of the series under consideration. Let $s_n = u_0 + u_1 + u_2 + \cdots + u_{n-1}$ to be the sum of the first n terms, where n denotes any integer number. If, for ever increasing values of n, the sum s_n indefinitely approaches a certain limit s, the series is said to be convergent, and the limit in question is called the sum of the series. On the contrary, if the sum s_n does not approach any fixed limit as it increases infinitely, the series is divergent, and does not have a sum. In either case, the term which corresponds to the index n, that is u_n, is what we call the general term. For the series to be completely determined, it is enough that we give its general term as a function of the index n." (*Bradley & Sandifer*, 2009). Clarity and naturalness of *Cauchy's definition** gives the impression that this is the only way we can define the sum of the infinite series and its convergence. It even seems to be the only available approach. The historical truth is different. First, we must realize that *infinite series* had already more than 2000 years of history at that time, and specific definition of an infinite series, with convergence defined in terms of ***limits of sequences*** of partial sums, was formulated later. Note that *Newton* used the term "*prime and ultimate ratio*" for the "*fluxion*", as the first or last ratio of two quantities just springing into being. *D'Alembert* replaced this notion by the *conception of a limit* in the article "*Limite*" of the "*Encyclopedia*"(edited by *D. Diderot* and until 1759 co-edited by *d'Alembert*) at the end of the 18th century. Moreover, as stated by *N. Bourbaki* (1993) *"And if d'Alembert is happier here, and recognises that in the "metaphysics" of the infinitesimal Calculus there is nothing other than the notion of limit (articles DIFFERENTIEL and LIMITE), he is no more able than his contemporaries, to understand the real meaning of expansion in divergent series, and to explain **the paradox of exact results obtained at the end of calculations with expressions deprived of any numerical interpretation**."* We

* Thus we currently define *a series* to be an ordered pair $(\{a_n\}, \{s_n\})$ of sequences connected by the relation $(s_n = \sum_{k=1}^{n} a_k)$ for all $n \in \mathbb{N}$.

can see that in some respect it looks like a task to solve a simple mathematical problem that can be found in *IQ tests*, but also in mathematical textbooks:

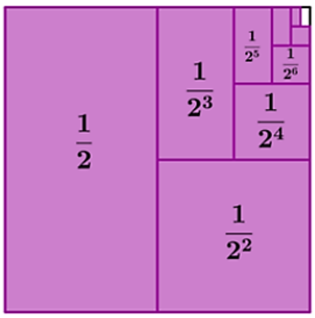
Find the next number in the sequence: $2, 4, 6, \dots$.

It is number 8 which is almost exclusively considered to be the natural and the only correct answer. Students are then very surprised (some authors of IQ tests may not know it now), that task has infinitely many solutions. Any real number $A \in \mathbb{R}$ is the solution of this problem. In other words, each (numerical) answer is correct. Just look at the problem in terms of numerical analysis, construct corresponding Lagrange polynomial a_n and determine a_4:

$a_n = 2\frac{(n-2)(n-3)(n-4)}{(1-2)(1-3)(1-4)} + 4\frac{(n-1)(n-3)(n-4)}{(2-1)(2-3)(2-4)} + 6\frac{(n-1)(n-2)(n-4)}{(3-1)(3-2)(3-4)} + A\frac{(n-1)(n-2)(n-3)}{(4-1)(4-2)(4-3)}, n \in \mathbb{N}$. We can easily verify that for $n = 4$ we really get $a_4 = A$.

Some historical remarks on infinite series

A time-honoured problem in this area is *Zeno of Elea's* paradoxes of "*Dichotomy*", and "*Achilles and the Tortoise*"*, which are concerned with convergent geometric series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots = 1$. According to *Aristotle,* Zeno's argument is a fallacy. For one cannot actually subdivide an interval infinitely many times. Infinite subdivision is only potential. In his *Physics* Aristotle (384-322 BC) himself implicitly underlined that the sum of a series of infinitely many addends (potentially considered) can be a finite quantity. Of course, it is possible to employ several visual representations (see *Figure 1*: the big square with sides long 1 unit, divided into a sequence of squares or triangles). In his *Quadratura parabolæ* Archimedes (287-212 BC) considered (implicitly, once again) a geometric series $1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots + \frac{1}{4^n} + \cdots = \frac{4}{3}$. Sums of other special geometric series were determined by mathematicians *Nicole Oresme* (1323 - 1382) and *R. Swineshead*. Geometric series played a crucial role in earlier research on series. *Swineshead* in his work (1350) when determining



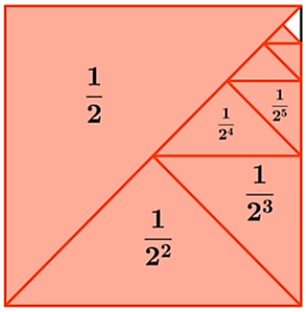


***Figure 5**:* Visualizations a sum of series $\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \cdots = 1$.

the average speed of uniformly accelerated motion needed to determine the sum of the first infinite series which was not geometric. He proved that $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n} + \cdots = 2$. *Oresme* was the first who managed to show (around 1350) that the harmonic series

**Dichotomy Paradox:* That which is in locomotion must arrive at the half-way stage before it arrives at the goal. *Achilles and the Tortoise:* In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead. (*Aristotle, Physics*)

$\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots$ is divergent. It is a very surprising result because the harmonic series diverges very slowly, e. g. the sum of the first 10^{43} terms is less than 100. The main *Oresme's* consideration consisted of an interesting estimate

$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\cdots+\frac{1}{8}\right)+\left(\frac{1}{9}+\cdots+\frac{1}{16}\right)+\cdots>1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots$. However, one must not conclude that *Oresme* or mathematicians in general began to distinguish convergent and divergent series. His results were lost several centuries, and the result was proved 1647 again by Italian mathematician *Pietro Mengoli* and in 1687 by Swiss mathematician *Johann Bernoulli*. In his *Varia Responsa* (1593) François Viète (1540 -1603) gave the formula for the sum of an infinite geometric progression. From Euclid´s *Elements* he took that the sum of n terms of $s_n=a_1+a_2+\cdots+a_n$ is given by $a_1:a_2=(s_n-a_n):(s_n-a_1)$. Then if $\mathrm{a}_1/\mathrm{a}_2>1$, a_n approaches 0 as becomes infinite, so that $s_\infty=\frac{(a_1)^2}{a_1-a_2}$ (*Kline*, 1972). A few decades later *Grégoire de Saint-Vincent* made geometric series a crucial instrument in his method of quadratures. Saint-Vincent, as well Viète, had an intuitive but clear idea of what the sum of series was (whatever words they used to denote the sum). Mengoli made a remarkable contribution to the uprising theory of series. Mengoli found $\sum_{k=1}^{\infty}\frac{(-1)^{n+1}}{n}=\ln 2$ and he showed how it is possible to determine the sum of several infinite series, which we now call the *telescopic series* (specifically $\sum_{k=1}^{\infty}\frac{1}{k(k+m)}$ for $m=1,2,3$). Another important step was made by Isaac Newton (1642 – 1727) in *A Treatise on the Methods of Series and Fluxions* (1671), when infinite series with numbers are extended to series containing variable expressions. Newton asserts that any proper operation that can be performed in arithmetic on numbers can likewise be performed in algebra on variable expressions. Just as arithmetic operations produce highly useful infinite decimal expressions, the same operations may produce highly useful infinite series in algebra. If we compare the following two processes (in modern symbols) that are almost identical, and in both cases the exception of the condition $q\neq 1$, *no restriction on q:*

1. If $s_n=a+aq+aq^2+\cdots+aq^{n-1}$, then $qs_n=aq+aq^2+\cdots+aq^{n-1}+aq^n$, therefore $qs_n-s_n=aq^n-a$, whence $s_n=a\frac{1-q^n}{1-q}, q\neq 1$.
2. If $S=a+aq+aq^2+aq^3+\cdots$, then $qS=aq+aq^2+aq^3+aq^4+\cdots$, therefore $S-qS=a$. Therefore, for $q\neq 1$ it holds $S=\frac{a}{1-q}$.

Using the previous relationships, we also get (for $q=x, q=-x$, $q=-x^2$) the following sums of infinite series: $1+x+x^2+x^3+\cdots=\frac{1}{1-x}, x\neq 1$, respectively $1-x+x^2-x^3+\cdots=\frac{1}{1+x}, x\neq -1, 1-x^2+x^4-x^6+\cdots=\frac{1}{1+x^2}$.

Newton, Euler and Lagrange considered infinite series to be *a part of algebra of polynomials*. It means that series were considered to be polynomials that can express the original function, without any *convergence control*.

Perhaps this is a point where most, if not all, students would agree and they would require no proof that there is anything wrong with this reasoning. From these geometric series a power series representation can be obtained for a wider variety of functions, since power series can be *differentiated* or *integrated term by term* in order to obtain a new power series. Of course in such a formal approach no restrictions on x do not occur. Isaac Newton in 1667 and *N. Mercator* in 1668 obtained the result $\sum_{k=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = \ln(1+x)$ by integrating the power series for $\frac{1}{1+x}$. The sensational discovery of sums of this series just opened quite new perspectives for the application of series, and mainly power series, to problems previously referred to as "impossible". J. Gregory in 1671 and G. W. Leibniz in 1673 obtained the result $\sum_{k=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4}$ by integrating the power series for $\frac{1}{1+x^2}$ and stating $x = 1$.

It is appropriate to note that according to Kline (1972) infinite series were in the 17th and 18th centuries and are still today considered to be the essential part of calculus. Indeed, Newton considered series inseparable from his method of fluxions because the only way he could handle even slightly complicated algebraic functions and the *transcendental functions* was to expand them into infinite series and differentiate or integrate term by term. Newton obtained many series for algebraic and transcendental functions. In his *De Analysi* in 1669 he provided the series for $\sin x, \cos x, \arcsin x, e^x$. The Bernoullis, Euler, and their contemporaries relied heavily on the use of series. Only gradually did the mathematicians learn to work with the *elementary functions in closed form*, that is, simple *analytical expressions*. Nevertheless, series were still the only representation for some functions and the most effective means of calculating the elementary transcendental functions. The successes obtained by using infinite series became numerous as the mathematicians gradually extended their discipline. The difficulties in the new concept were not recognized, at least for a while. Series were just infinite polynomials and appeared to be treatable as such. Moreover, it seemed clear, as Euler and Lagrange believed, that every function could be *expressed in form of a series*.

Intuitive understanding of the concept of the sum of an infinite series and often mechanical transmission of properties of finite sums on infinite sums has brought many problems and paradoxical outcome. Simply said, a finite sum is well-defined, an infinite sum is not. This can be illustrated by a simple example of infinite series.

Let s denote the sum of (telescopic) convergent series $s = \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots$. Then $s = \left(\frac{1}{1} - \frac{2}{3}\right) + \left(\frac{2}{3} - \frac{3}{5}\right) + \left(\frac{3}{5} - \frac{4}{7}\right) + \cdots = \frac{1}{1} - \frac{2}{3} + \frac{2}{3} - \frac{3}{5} + \frac{3}{5} - \frac{4}{7} + \cdots = 1$, since all terms after the first one are mutually cancelled out. Again $\mathrm{s} = \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right) + \cdots = \frac{1}{2} - \frac{1}{6} + \frac{1}{6} - \frac{1}{10} + \frac{1}{10} - \frac{1}{14} + \cdots = \frac{1}{2}$, since all terms after the first are cancelled out. Then $1 = \frac{1}{2}$.

The series which provoked the greatest debates and controversy is $1 - 1 + 1 - 1 + \cdots + (-1)^{n+1} + \cdots$. It seemed clear that by writing the series as $(1-1) + (1-1) + (1-1) + \cdots$ the sum should be 0. It also seemed clear that by writing the series as $1 + (-1+1) + (-1+1) + (-1+1) + \cdots$ the sum should be 1. However, if S is used to denote sum $S = 1 - 1 + 1 - 1 + \cdots$, then $S = 1 - (1 - 1 + 1 - 1 + \cdots)$, i. e. $S = 1 - S$, so $S = \frac{1}{2}$. Guido Grandi (1671 - 1742), a professor of mathematics at University of Pisa, in his book *Quadratura Circuli et Hyperbolae* (1703), obtained the result by another method. He set $x = 1$ in the expansion $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$ and obtained $\frac{1}{2} = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$. *Guido Grandi* considered the formula $\frac{1}{2} = 1 - 1 + 1 - 1 + 1 - 1 + \cdots = (1-1) + (1-1) + \cdots = 0 + 0 + 0 + \cdots = 0$, to be the symbol for creation of world from *Nothing*. He obtained the result $\frac{1}{2}$ by considering the case of a father who bequeathed a gem to his two sons, each allowed to keep it one year alternately. It then belonged to each son by one half. (*Struik*, 1948). Several mathematicians of this period (*J. Riccati, P. Varignon* and *Nicholas I. Bernoulli*) did not agree with this reasoning. The same result, based on interesting probabilistic reasoning, was arrived at by G. W. Leibniz. Instead, Leibniz argued that if one takes the first term, the sum of the first two, the sum of the first three, and so forth, one obtains $1, 0, 1, 0, 1, \ldots$. Thus 1 and 0 are equally probable; one should therefore take the arithmetic mean, which is also the most probable value, as the sum. This solution was accepted by James and John Bernoulli, Daniel Bernoulli, and Lagrange. Leibniz conceded that "his argument was more metaphysical than mathematical, but went on to say that there was more metaphysical truth in mathematics than was generally recognized" (*Kline*, 1972, p. 446). *Christian Wolf* (1678 -1754) wished to conclude that $1 - 2 + 4 - 8 + \cdots = \frac{1}{3}$, $1 - 3 + 9 - 27 + \cdots = 1/4$ by using an extension of Leibniz's own probability argument. Real extensive work on series began about 1730 with *Leonhard Euler* (1707-1783), who aroused tremendous interest in the subject. *Euler* summarizes Leibniz's arguments and he wrote: "Now if, therefore, the series is taken to infinity and (consequently) the number of terms cannot be regarded as either even or odd, it cannot be concluded that the sum is either 0 or 1, but we ought to take a certain median value which differs equally from both, namely ½." To obtain the sum of $1 - 1 + 1 - 1 + \cdots$. His second argument pointed out in his textbook *Institutiones calculi differentialis* (1755):

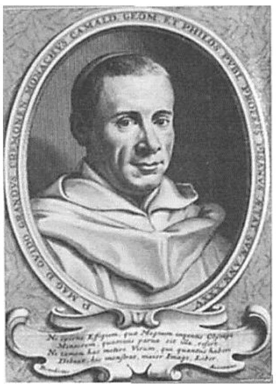


***Figure 6**:* Luigi Quido Grandi (1671 -1742)

***Figure 7**:* Leonhard Euler (1707 - 1783)

"We state that the sum of an infinite series is the finite expression by which the series is generated. From this point of view the sum of the infinite series $1 - x + x^2 - x^3 + \cdots$ is $1/(1 + x)$ because the series arises from the development of the fraction, for every value x."

As divergent series are considered such an inconvenience, we can settle defining the sum of a series in terms of limits of sequences of partial sums (as we do at present) and dismiss as "divergent" any series that does not satisfy this convergence requirement. ***Well, the lecturer, who presented at the Conference in Nitra, knew well Cauchy´s definition of sum of infinite series.*** Another alternative is to redefine the concept of "sum". Among those making the attempt to save the *divergent series* for analysis was Leonhard Euler. He attempted to redefine the meaning of "sum" in a significantly more abstract fashion, further from the then common understanding of "*sum*" as "*to add up*." Euler proposed the following definition for "sum", also referred *Euler's principle*: *"Understanding of the question is to be sought in the word "sum"; this idea, if thus conceived-namely, the sum of a series is said to be that quantity to which it is brought closer as more terms of the series are taken-has relevance only for convergent series, and we should in general give up this idea of sum for divergent series. Wherefore, those who thus define a sum cannot be blamed if they claim they are unable to assign a sum to a series. On the other hand, as series in analysis arise from the expansion of fractions or irrational quantities or even of transcendentals, it will in turn be permissible in calculations to substitute in place of such a series that quantity out of whose development it is produced. For this reason, if we employ this definition of sum, that is, to say the sum of a series is that quantity which generates the series, all doubts with respect to divergent series vanish and no further controversy remains on this score, inasmuch as this definition is applicable equally to convergent or divergent series."* (*Barbeau and Leah*, 1976).

The intuitive formation of this definition of "sum" reflects an attitude still current among applied mathematicians and physicists: problems that arise naturally (i.e., from nature) do have solutions, so the assumption that things will work out eventually is justified experimentally without the need for existence sorts of proof. Assume everything is okay, and if the arrived-at solution works, you were probably right, or at least right enough. (Lehmann, 2000). For example *Euler* wrote

$$1/4 = 1 - 2 + 3 - 4 + \cdots, \qquad 0 = 1 - 3 + 5 - 7 + \cdots, \qquad -1 = 1 + 2 + 4 + 8 + \cdots$$

because these series arose from expansions

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \cdots, \frac{1-x}{(1+x)^2} = 1 - 3x + 5 - 7x^3 + \cdots, \frac{1}{1-2x} = 1 + 2x + 4x^2 + \cdots$$

and setting $x = 1$.

N. Bernoulli replied that the same series might arise from expansion of two different functions and, if so, the sum would not be unique. From a theoretical point of view this would be a serious problem. And, really *J. Ch. Callet* showed that the series $1 - 1 + 1 - 1 + \cdots$ may be obtained from the expansion

$\frac{1+x}{1+x+x^2} = \frac{1-x^2}{1-x^3} = 1 - x^2 + x^3 - x^5 + x^6 - x^8 + \cdots$ and setting $x = 1$, we get $\frac{2}{3}$ instead Euler's $\frac{1}{2}$. *Joseph-Louis L. Lagrange* (1736 -1813) considered this objection and argued that Callet's example was incomplete. When the missing terms were included, the series should have been written

$$1 + 0x - x^2 + x^3 + 0x^4 - x^5 + x^6 + 0x^7 - x^8 + \cdots$$

so what was summed was $1 + 0 - 1 + 1 + 0 - 1 + 1 + 0 - 1 + \cdots$ a series whose partial sums are $1, 1, 0, 1, 1, 0, \ldots$ with average sums $\frac{2}{3}$. And, in fact, Euler's assertion, when properly interpreted, is correct, since a convergent power series has a unique generating function. It is clear that before the 19th century divergent series were widely used by Euler and others, but often led to confusing and contradictory results. English mathematician *G. H. Hardy* (1877-1947), author of the excellent book *Divergent series* (1949) suggests that “*...It is a mistake to think of Euler as a 'loose' mathematician, though his language may sometimes seem loose to modern ears; and even his language sometimes suggests a point of view far in advance of the general ideas of his time. …Here, as elsewhere, Euler was substantially right. The puzzles of the time about divergent series arose mostly, not from any particular mystery in divergent series as such, but from disinclination to give formal definitions and from the inadequacy of the current theory of functions. It is impossible to state Euler's principle accurately without clear ideas about functions of a complex variable and analytic continuation*”. It must, however, admit that in spite of great Euler’s authority divergent series arouse ever greater mistrust. This attitude of mathematicians is succinctly explained by Norwegian mathematician *Niels H. Abel* (1802 – 1829) in 1828: “*Divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever*.“ In the ensuing period of critical revision they were simply rejected. Then came a time when it was found that something after all could be done about them. Mathematics after Euler moved slowly but steadily towards the orthodoxy ultimately imposed on it by *Cauchy*, *Abel*, and their successors, and divergent series were gradually banished from analysis, to appear only in quite modern times. After *Cauchy*, the opposition seemed definitely to have won.

The above mentioned information should not lead to the impression that Cauchy's *Cours d'Analyse* textbook, which already contains the foundations of his new theory of infinite series, was a step backwards. The opposite is true. It was a very necessary and important step. *Cauchy* was in fact another brilliant mathematician who greatly influenced the character of infinitesimal calculus. Victory of the Cauchy's approach was very important for the further development of the actual theory of infinite series. Cauchy's formulation of the definition of the sum of the infinite series by a sequence of partial sums limits and precise distinction between convergent and divergent series, was a ground-breaking milestone in the history of the theory of series. This has become the standard for the next period, providing a uniform platform and some unified approach for justifying and derivation of results.

Cauchy's attitude to divergent series is openly declared in the introduction to the Cours d'Analyse: “*As for methods, I have sought to give them all the rigor that one requires in geometry, so as never to have recourse to the reasons drawn from the generality of algebra. Reasons of this kind, although commonly admitted, particularly in the passage from convergent series to divergent series, and from real quantities to imaginary expressions, can, it seems to me, only sometimes be considered as inductions suitable for presenting the truth, but which are little suited to the exactitude so vaunted in the mathematical sciences. We must at the same time observe that they tend to attribute an indefinite extension to algebraic formulas, whereas in reality the larger part of these formulas exist only under certain conditions and for certain values of the quantities that they contain. Determining these conditions and these values, and fixing in a precise way the sense of the notations I use, I make any uncertainty vanish; and then the different formulas involve nothing more than relations* *among real quantities, relations which are always easy to verify on substituting numbers for the quantities themselves. In order to remain faithful to these principles, I admit that I was forced to accept several propositions which seem slightly hard at first sight. For example . . . a divergent series has no sum*." (*Cauchy*, 1821, ii–iii). The initial reaction of our founders of nineteenth-century analysis (Cauchy, Abel, and others) was that valid arguments could be based only on convergent series. Divergent series were mostly excluded from mathematics. We saw that many eighteenth century mathematicians achieved spectacular results with divergent series but without a proper understanding of what they were doing. They reappeared in 1886 with *Poincare's* work on asymptotic series. In 1890 *Ernesto Cesàro* (1859-1906) realized that one could give a rigorous definition of the sum of some divergent series, and defined *Cesàro summation* as follows:

If $s_n = a_1 + a_2 + \cdots + a_n$ and $\lim_{n\to\infty} \frac{s_1+s_2+\cdots+s_n}{n} := s \in \mathbb{R}$, then we call s the $(C,1)$ sum of $\sum_{n=1}^{\infty} a_n$ and we write $\sum_{n=1}^{\infty} a_n = s[\text{Cesàro}]$.

For example $\sum_{n=1}^{\infty}(-1)^{n+1} = \frac{1}{2}[\text{Cesàro}]$, but $\sum_{n=1}^{\infty} n$ is not Cesàro summable because the terms of the sequence of means of partial sums $\{t_n\}$, $t_n = \frac{1}{n}\sum_{k=1}^{n} s_k$ are here $\frac{1}{1}, \frac{4}{2}, \frac{10}{3}, \frac{20}{4}, \ldots$ and this sequence diverges to infinity. Cesàro's key contribution was not the discovery of this method but his idea that one should give an explicit definition of the sum of a divergent series. In the years after Cesàro's paper several other mathematicians gave other definitions of the sum of a divergent series, though these are not always compatible: different definitions can give different answers for the sum of the same divergent series, so when talking about the sum of a divergent series it is necessary to specify which summation method one is using. In addition to Cesàro summation to the most known summation methods include Abel summation and Euler summation. For illustration we add Abel summation method which is similar to the well-known *Abel's theorem* on power series:

If $\sum_{n=1}^{\infty} a_n x^n$ is convergent for $0 \le x < 1$ (and so for all x, with $|x| < 1$), $f(x)$ is its sum, and $\lim_{x\to 1-0} f(x) = s$, then we call s the $A\,sum$ of $\sum_{n=1}^{\infty} a_n x^n$ and we write $\sum_{n=1}^{\infty} a_n = s[\text{Abel}]$.

Abel summation is interesting in part because it is consistent with and at the same time more powerful than *Cesàro summation.* We say that *asummability method* M is *regular* if it is equal to the commonly used limit (of partial sums) on all convergent series. It can be easily verified that the *Cesàro summation* and *Abel summation are regular methods.* Such a result is called an abelian theorem for M, from the prototypical Abel's theorem. More interesting and in general more subtle are partial converse results, called *tauberian theorems*, from a prototype proved by Austrian mathematician *Alfred Tauber**. Here partial converse means that if M sums the series Σ, and some side-condition holds, then Σ was convergent in the first place; without any side condition such a result would say that M only summed convergent series (making it useless as a summation method for divergent series). Finally, let us note,

* Tauber was born in Pressburg, now *Bratislava, Slovakia* in 1866 and died in concentration camp in Theresienstadt, now *Terezin, Czech Republic* in around 1942.

using *Hanh-Banach theorem* which is a central tool in functional analysis, ***it can be shown that each series whose sequence of partial sums is bounded is summable through some methods.*** Unfortunately proof of this statement is non-constructive. Mathematicians introduced recurrent series and emphasized the law of formation of coefficients, independent of the convergence of series. The attempt to increase the speed of convergence of series subsequently led to the emergence of asymptotic series, which showed the possibility of using divergent series to obtain appropriate approximations.

In conclusion, let us get back to the infinite series $\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + \cdots$ which we started our considerations with. This series belongs to infinite series which are assumed sums only with tough efforts, since this series is neither *Cesàro summable*, nor *Abel summable.* Using the equation $\sum_{n=1}^{\infty}(-1)^{n+1} n x^n = \frac{1}{(1+x)^2}$, it can be rigorously proved that

$$\sum_{n=1}^{\infty}(-1)^{n+1} n = \lim_{x\to 1^-} \sum_{n=1}^{\infty}(-1)^{n+1} n x^n = \lim_{x\to 1^-} \frac{1}{(1+x)^2} = \frac{1}{4}\,[\text{Abel}].$$

Euler then went on to compute the sum of all natural numbers, as follows. First, he considered what is now called the *Riemann zeta function*[*]:

$$\zeta(s) = 1^{-s} + 2^{-s} + 3^{-s} + 4^{-s} + 5^{-s} + 6^{-s} + \cdots.$$

Multiplying by 2^{1-s}, he obtained $2^{1-s}\zeta(s) = \quad 2.2^{-s} + \quad 2.4^{-s} + \quad 2.6^{-s} + \cdots.$ Subtracting the second equation form the first one, he got

$$(1 - 2^{1-s})\zeta(s) = 1^{-s} - 2^{-s} + 3^{-s} - 4^{-s} + 5^{-s} - 6^{-s} + \cdots,$$

then, after evaluating both sides at $s = -1$, he got

$$\zeta(-1) = 1 + 2 + 3 + 4 + \cdots = \left(-\frac{1}{3}\right)(1^{-s} - 2^{-s} + 3^{-s} - 4^{-s} + 5^{-s} - 6^{-s} + \cdots) = -\frac{1}{12}.$$

Conclusions

History of mathematics can deepen the conceptual understanding of mathematical concepts and theories and lighten their roots and sources. This understanding is essential in education as well as in further practice and support for good decision-making. In our contribution we present one basic example of the use of history of mathematics to help lecturers as well as learners understand and overcome epistemological obstacles in the development of mathematical understanding of series. Based on the principal that "*ontogeny recapitulates phylogeny*" – we see it appropriate that the development of an individual's mathematical understanding respect the historical development of mathematical ideas. Even though this approach is demanding in many ways we see this combination with historical and psychological perspectives very promising in further teacher development.

We presented this approach with a short overview of history of series theory. We would like to stress that divergent series should be treated with the same respect as convergent series. The first course in series methods often gives the impression of obsession with the issue of convergence or divergence of a series. The huge amount of tests might lead one to this conclusion. Accordingly, you may have decided that convergent series are useful and proper tools of analysis while divergent series are useless and without merit. In fact divergent series are, in many instances, as important as, or more important than convergent ones. Many eighteenth-century mathematicians achieved spectacular results with divergent series but

[**] The *Riemann zeta function* is an extremely important special function of mathematics and physics that arises in definite integration and is intimately related with very deep results surrounding the prime number theorem. While many of the properties of this function have been investigated, there remain important fundamental conjectures (most notably the *Riemann hypothesis*) that remain unproved to this day.

without proper understanding of what they were doing. The initial reaction of the founders of the nineteenth-century analysis (Cauchy, Abel, and others) was that valid arguments could be based only on convergent series, and that divergent series should be avoided. There are many useful ways of doing rigorous work with divergent series. One way, which we now study, is the development of summability methods.

This approach also needs to be more emphasized during prospective teacher preparation and teachers should transfer the approach into their teaching.

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