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Pencils of Planes and Spheres in Problem Solving

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Abstract

In the present paper we present some facts in the theory of a pencil of planes and spheres which can be defined as a linear combination of their equations. In addition, we discuss possible approach how to use the pencils of planes or pencil of spheres in solving some problems in the analytic geometry.

Keywords: Pencil of planes, pencil of spheres, linear combination.

Classification: G44, D44

Introduction

The notion of linear combination in problem solving within analytic geometry is not a common one. Most often, we use a linear combination as related to a pencil of lines or of circles in Euclidean plane (see [1], [2]). Like in the case of pencils of planar curves, a notion of a pencil of planes and spheres is very useful, and we define it as follows. A pencil of planes or of spheres is the family of all planes or of spheres in space which pass through the intersection of two fixed planes or the intersection of a sphere and a plane or two spheres. In other words, a pencil of planes (spheres) is the family of planes (spheres) through a fixed line (circle). In this article, we look at some pencils in three dimensional Euclidean space, and we demonstrate a method in which pencils are used as tools in problem solving.

Pencil of planes

Let consider two planes P_1 and P_2 having a common line of intersection, given by their equations:

$$P_1 : a_1x + b_1y + c_1z + d_1 = 0 \tag{1}$$

$$P_2 : a_2x + b_2y + c_2z + d_2 = 0 \tag{2}$$

If we multiply an equation (1) by α and (2) by β, where α and β are real parameters not both equal to zero, and we add both equations, we have

$$\alpha(a_1x + b_1y + c_1z + d_1) + \beta(a_2x + b_2y + c_2z + d_2) = 0 \tag{3}$$

i.e. we have a *linear combination* of equations (1) and (2). From the condition of existence of the non-empty intersection of planes P_1 and P_2 follows a linearly independence of the normal vectors (a_1, b_1, c_1), and (a_2, b_2, c_2) to these planes. The set of all points of 3-

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dimensional Euclidean space the coordinates of which satisfy equation (3) is a plane that contains common points of plane P_1 and P_2, because their coordinates satisfy equations (1) and (2) simultaneously. In other words, linear combination (3) is the equation of pencil of planes that contain the unique intersection line of P_1 and P_2.

Problem 1. Consider the line L of intersection of the two planes given by

$$x+y-3=0, \quad x-2y+z=0.$$

Also, consider the plane $P: 2x+z+1=0$ (Figure 1). Find an equation of plane containing the line L and perpendicular to the plane P.

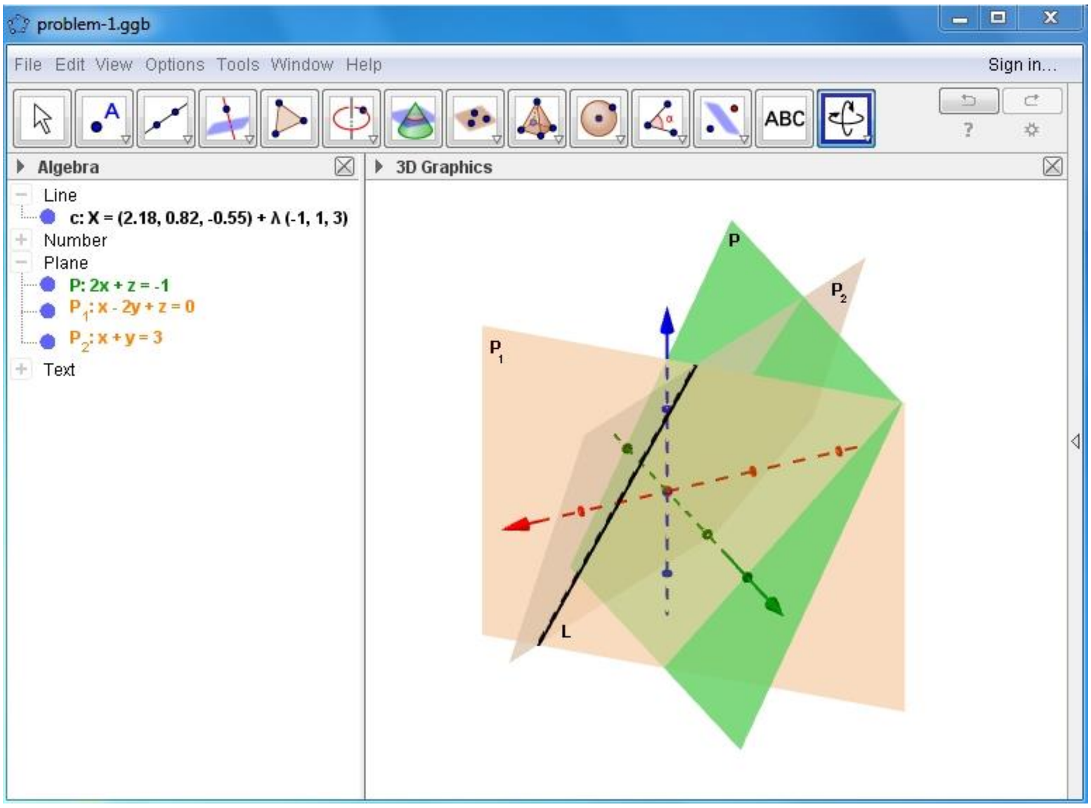

Figure 1

Solution. The plane is determined by equation (3) in the form

$$\alpha(x+y-3)+\beta(x-2y+z)=0 \text{ or } (\alpha+\beta)x+(\alpha-2\beta)y+\beta z-3\alpha=0.$$

The normal vector to the plane is perpendicular to the plane P, hence their dot product is

$$2.(\alpha+\beta)+0.(\alpha-2\beta)+1.\beta=0.$$

This is equivalent to $2\alpha=-3\beta$. If we set $\alpha=3$, then $\beta=-2$, and the plane we are looking for has equation

$$x+7y-2z-9=0.$$

Pencil of spheres

Let us consider the plane P, and the sphere S defined by the following equations:

$$P: ax+by+cz+d=0 \tag{4}$$

$$S:(x-c_1)^2+(y-c_2)^2+(z-c_3)^2-r^2=0 \tag{5}$$

If we multiply an equation (4) by a real number α and sum both equations, we have

$$\alpha(ax+by+cz+d)+[(x-c_1)^2+(y-c_2)^2+(z-c_3)^2-r^2]=0 \tag{6}$$

This is equivalent to the following equation

$$(x-A)^2+(y-B)^2+(z-C)^2=A^2+B^2+C^2-D \tag{7}$$

where

$$A=c_1-\frac{\alpha}{2}a,\ B=c_2-\frac{\alpha}{2}b,\ C=c_3-\frac{\alpha}{2}c,\ D=c_1^2+c_2^2+c_3^2-r^2.$$

What can we say about the set of all points in three dimensional Euclidean space, which coordinates satisfy (7)?

If $A^2+B^2+C^2-D<0$ then there doesn't exist a point the coordinates of which satisfy equation (6). Otherwise it can be a point or a sphere. If the sphere S and the plane P have at least two common points then (7) is the equation of a sphere which contains all of their common points, because their coordinates satisfy the equations (4) and (5).

Problem 2. Let C be a circle given as the intersection of plane P and sphere S with the equations

$$P: x-y-z+2=0,\quad S:(x-1)^2+(y+2)^2+z^2-16=0.$$

Find the equation of a sphere which contains the circle C and passing through the point $M[2,-3,2]$.

Solution. The equation of our sphere has the form

$$\alpha(x-y-z+2)+[(x-1)^2+(y+2)^2+z^2-16]=0$$

As it contains the point M, its coordinates $[2,-3,2]$ satisfy the above equation, and we have $\alpha=2,$ and the sphere we are looking for has equation

$$x^2+(y+1)^2+(z-1)^2=9.$$

Problem 3. Let C be a circle given as the intersection of plane P and sphere S with the equations

$$P: x-y-z+2=0,\quad S:(x-1)^2+(y+2)^2+z^2-18=0.$$

Also, consider the line $L: y-3=0, 2y+z-5=0.$ Find the equation of a sphere which contains the circle C and tangent to the line $L.$

Solution. The equation of the sphere in question has the form

$$\alpha(x-y-z+2)+[(x-1)^2+(y+2)^2+z^2-18]=0.$$

Every point on the line L has coordinates $[t,3,-1], t\in R.$ Substituting these coordinates into the above equation, we have the quadratic equation with parameter α in the form

$$t^2+(\alpha-2)t+9=0.$$

The solution of this equation are x-coordinates of the common points of our sphere and given line L. Because L is the line tangent to the sphere, discriminant of the quadratic equation is equal to zero, i.e. $(\alpha-2)^2-4.1.9=0$. From this it is obvious that $\alpha=8$, or $\alpha=-4$. Therefore, there are two such spheres we are looking for (Figure 2):

$$(x+3)^2+(y-2)^2+(z-4)^2=26 \text{ and } (x-3)^2+(y+4)^2+(z+2)^2=50.$$

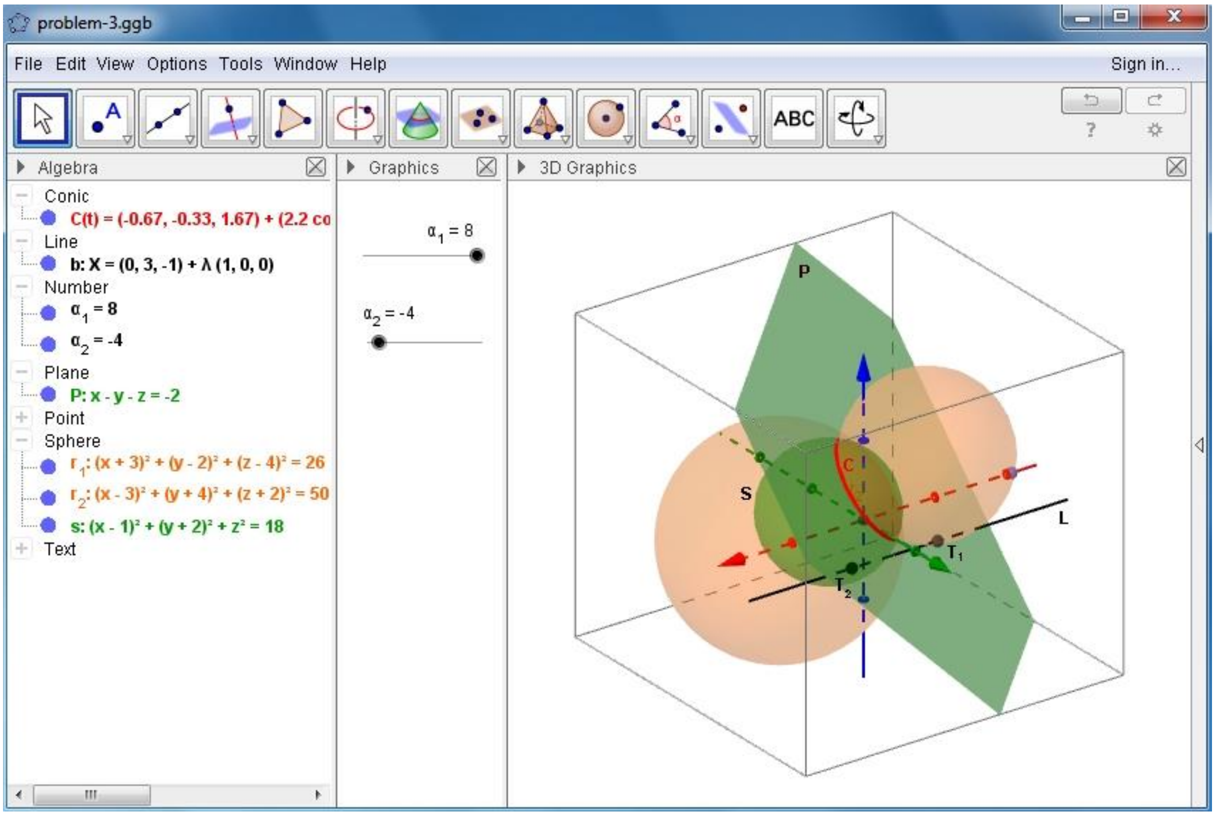

Figure 2

Now, we consider two non-concentric spheres with the equations:

$$S_1:(x-m_1)^2+(y-m_2)^2+(z-m_3)^2-r_1^2=0 \tag{8}$$

$$S_2:(x-n_1)^2+(y-n_2)^2+(z-n_3)^2-r_2^2=0 \tag{9}$$

If we multiply an equation (8) by α and (9) by β, where α and β are real parameters not both equal to zero, and if we add both equations, we obtain

$$(\alpha+\beta)(x^2+y^2+z^2)-2[(\alpha m_1+\beta n_1)x+(\alpha m_2+\beta n_2)y+(\alpha m_3+\beta n_3)z]+$$
$$\alpha m_1^2+\beta n_1^2+\alpha m_2^2+\beta n_2^2+\alpha m_3^2+\beta n_3^2-\alpha r_1^2-\beta r_2^2=0 \tag{10}$$

What can we say about the set of all points in space, the coordinates of which satisfy (10)?

If $\alpha+\beta=0$, then without loss of generality, we can set $\alpha=1$, $\beta=-1$, and (10) is the equation of the plane

$$2(n_1-m_1)x+2(n_2-m_2)y+2(n_3-m_3)z+(m_1^2+m_2^2+m_3^2-r_1^2)-(n_1^2+n_2^2+n_3^2-r_2^2)=0 \tag{11}$$

Given spheres are non-concentric, therefore at least one of the coefficients $(m_i-n_i), i=1,2,3$ is not equal to zero. This plane is perpendicular to the line of centers of both given spheres.

It will be called the *radical plane* of all the spheres of pencil determined by given spheres S_1 and S_2.

If $\alpha+\beta\neq 0,$ then we could write equation (10) in the form

$$(x-A)^2+(y-B)^2+(z-C)^2=A^2+B^2+C^2-D \qquad (12)$$

where

$$A=\frac{\alpha m_1+\beta n_1}{\alpha+\beta},\ B=\frac{\alpha m_2+\beta n_2}{\alpha+\beta},\ C=\frac{\alpha m_3+\beta n_3}{\alpha+\beta},\ D=\frac{\alpha(m_1^2+m_2^2+m_3^2)+\beta(n_1^2+n_2^2+n_3^2)}{\alpha+\beta}$$

A number of the common points of these spheres depends on the right side of equation (12). Denote $R=A^2+B^2+C^2-D.$

- *(i)* If $R<0,$ then there doesn't exist a point satisfying (12);
- *(ii)* If $R=0,$ then only the point $[A,B,C]$ satisfies (12);
- *(iii)* If $R>0,$ then equation (12) determines the sphere with center $[A,B,C]$ and radius $\sqrt{R}.$

The *pencil* determined by spheres S_1 and S_2 is the set of all spheres with equations of the form (10). If the spheres intersect at least in two points, the radical plane contains their circle of intersection. A distance from the center $M[m_1,m_2,m_3]$ of sphere S_1 to the radical plane is given by

$$d=\frac{|\,2(n_1-m_1)m_1+2(n_2-m_2)m_2+2(n_3-m_3)m_3+(m_1^2+m_2^2+m_3^2-r_1^2)-(n_1^2+n_2^2+n_3^2-r_2^2)\,|}{\sqrt{[2(n_1-m_1)]^2+[2(n_2-m_2)]^2+[2(n_3-m_3)]^2}}$$

which simplifies to

$$d=\frac{|\,(n_1-m_1)^2+(n_2-m_2)^2+(n_3-m_3)^2+r_1^2-r_2^2)\,|}{2\sqrt{(n_1-m_1)^2+(n_2-m_2)^2+(n_3-m_3)^2}}.$$

a) If $d>r_1,$ then the intersection of given spheres be an empty set;

b) If $d=r_1,$ then the intersection of given spheres be only one point P (as a point of intersection of the line through the centeres of the given spheres and the radical plane);

c) If $d<r_1,$ then the intersection of given spheres be a circle in the radical plane, with radius $\sqrt{r_1^2-d^2}$ and the center $P.$

Problem 4. Determine the equation of a sphere through the circle of intersection of two given spheres S_1 and S_2 and tangent to the plane $P,$ where

$$S_1: x^2+y^2+z^2=9,\ \ S_2:(x-1)^2+(y+1)^2+(z-2)^2=7,\ \ P: x-5=0.$$

Solution. The circle we are looking for has the equation

$$\alpha(x^2+y^2+z^2-9)+\beta[(x-1)^2+(y+1)^2+(z-2)^2-7]=0,\ \ \alpha+\beta\neq 0.$$

By substituting $x=5$ (from the equation of P) into this equation we have

$$(\alpha+\beta)y^2+2\beta y+(\alpha+\beta)z^2-4\beta z+16\alpha+14\beta=0.$$

which simplifies to

$$(y-A)^2+(z-B)^2=A^2+B^2-C, \text{ and } A=\frac{\beta}{\alpha+\beta},\ B=\frac{2\beta}{\alpha+\beta},\ C=\frac{16\alpha+14\beta}{\alpha+\beta}.$$

As the sphere to be found should be tangent to the plane P, the right side A^2+B^2-C of above equation will be equal to zero. We get

$$5\beta^2=(\alpha+\beta)(16\alpha+14\beta) \quad \Rightarrow \quad \left(\frac{8\alpha}{\beta}+3\right)\left(\frac{2\alpha}{\beta}+3\right)=0.$$

If we set $\alpha=-3$ or $\alpha=3$ then $\beta=8$ or $\beta=-2$, and the spheres have the following equations:

$$\left(x-\frac{8}{5}\right)^2+\left(y+\frac{8}{5}\right)^2+\left(z-\frac{16}{5}\right)^2=\frac{289}{25} \text{ and } (x+2)^2+(y-2)^2+(z+4)^2=49.$$

Conclusion

We can see that the pencils of planes or of spheres may be useful for solving some problems in analytic geometry in 3-dimensional Euclidean space. In addition, there exist dynamic geometry environments like GeoGebra to visualize both planes and spheres, and also the linear combination of their equations. Although the examples given in this paper are related to the planes and spheres, the method of pencils is also applicable to other surfaces in space.

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