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Extensions of Cascades Created by Certain Function Systems

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Abstract

Considering three simple function systems which consist of power functions with odd exponents and two linear functions of one real variable, we are constructing actions of the additive group of all integers on the set of all real numbers, i.e. cascades. Using certain extensions based on prolongations of flows, we obtain the system of cascades which are mutually isomorphic. One from consequences of the result is that all solution sets of corresponding functional equations formed with the use of given functions are non-empty, moreover all solution sets consisting of permutations of the set of all reals are non-empty, as well.

Keywords: Mono–unary algebra, orbital decomposition, cascade.

Classification: 26A09, 37B05

Introduction

The theory of dynamical systems is a very broad mathematical area historically arising from the theory of ordinary differential equations. As it has been mentioned in [1, 2], the qualitative theory of ordinary differential equations and the theory of dynamical systems arose within the theory of differential equations; in time, the theory of dynamical systems attained a definite autonomy, and it can now be regarded as an independent branch of mathematics, which continues to develop intensively. It retains a close connection with the theory of differential equations, and the boundary between them is not particularly sharp. At the same time, the theory of dynamical systems has established new connections with the branches of mathematics which appear ever more essential for certain questions in the theory of dynamical systems. Even the concept of a dynamical system has itself evolved considerably – [2, 3, 4, 7, 12].

In this contribution we will concentrate ourselves onto cascades (called also discrete flows). Let us recall that a flow is in fact a one-parameter group or semigroup of transformations acting on a set M, which is called the phase space of the flow. In other words, associated to each $t \in \mathbf{R}$ (the set of all real numbers) or $t \in \mathbf{R}_0^+$ (the set of all non-negative real numbers), there is a mapping $g^t : M \to M$ such that the group property holds, i. e.

$$g_0 = id_M,\ g^{t+s} = g^t \circ g^s,$$

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for all *t, s* under consideration. A cascade differs from a flow in that the maps g^t are only defined for $t \in \boldsymbol{Z}$ (the set of all integers) or $t \in \boldsymbol{N_0}$ (the set of all non-negative integers).

If $k \in \boldsymbol{Z}$, the notation g^k denotes the k-th iterate of the mapping $g = g^1$ for $k > 0$ and the k-th iterate of g^{-1} when $k < 0$. The name „cascade“ is used to contrast it to a „flow“. Flows are most often encountered in applications, but cascades also appear. For example, in ecology one might want to study changes in a population with non-overlapping adult generations. Here the generations play the role of discrete time. Nevertheless, the main significance of cascades lies in the fact that they are usually technically somewhat simpler than flows; at the same time, the essence of the matter may be the same in both cases. Thus, results obtained for cascades frequently carry over to flows, often not by way of a formal reduction, but by some modifications of the proofs.

Preliminaries

Let us recall that a cascade is sometimes called a discrete flow or a flow with a discrete time. Let us mention other necessary concepts. Functions $f: \boldsymbol{R} \to \boldsymbol{R}$, $g: \boldsymbol{R} \to \boldsymbol{R}$ are called conjugated if there is a bijection $h: \boldsymbol{R} \to \boldsymbol{R}$ such that $h \circ f = g \circ h$ [10]. Corresponding mono-unary algebras $(\boldsymbol{R}, f)$, $(\boldsymbol{R}, g)$ are then said to be isomorphic and the mapping h is an isomorphism. In general, a mapping $h: (\boldsymbol{R}, f) \to (\boldsymbol{R}, g)$ with the above property $(h(f(x)) = g(h(x)), x \in \boldsymbol{R})$, is called a homomorphism of the mono-unary algebra $(\boldsymbol{R}, f)$ into the mono-unary algebra $(\boldsymbol{R}, g)$ - [9].

Let $(G, \cdot, e)$ be a group with the unit e, $X \neq \phi$, $\Theta: X \times G \to X$ be a mapping satisfying these conditions:

(i) $\Theta(x, e) = x$ for any $x \in X$ (the Identity axiom),

(ii) $\Theta(\Theta(x, a), b) = \Theta(x, a \cdot b)$ for all $x \in X$, $a, b \in G$ (the Homomorphism axiom or the Mixed associability condition – MAC).

Then the triad $\boldsymbol{A} = (X, G, \Theta)$ is said to be the action of the group G on the set X [6] or a discrete dynamical system with the phase group G and the phase set (space) X [12]. If $G = \boldsymbol{R}$, X is a metric space or a topological space, $\Theta: X \times G \to X$ is a continuous mapping, then the action $\boldsymbol{A}$ is called a flow. If $G = (\boldsymbol{Z}, +)$ and X is a set without – not necessary – any additional structure, the action $\boldsymbol{A}$ is termed a cascade.

If $\boldsymbol{A} = (X, G, \Theta_A)$, $\boldsymbol{B} = (Y, G, \Theta_B)$ are cascades with the same phase group G, then a mapping $h: X \to Y$ such that $\Theta_B(h(x), a) = h(\Theta_A(x, a))$ for any pair $[x, a] \in X \times G$ is said to be a homomorphism of the cascade $\boldsymbol{A}$ into the cascade $\boldsymbol{B}$. If h is a bijection, then this homomorphism is called an isomorphism. If some isomorphism $h: \boldsymbol{A} \to \boldsymbol{B}$ exists, the cascades $\boldsymbol{A}$, $\boldsymbol{B}$ are said to be isomorphic and we write $\boldsymbol{A} \cong \boldsymbol{B}$.

Let (X, f) be a mono-unary algebra. Let us denote by $f^n: X \to X$ the n-th iteration of the mapping f. Let us suppose $f: X \to X$ is a bijective mapping. To the algebra (X, f) we can assign the action $\boldsymbol{A_f} = (\boldsymbol{Z}, X, \Theta_f)$, where $\Theta_f(m, x) = f^m(x)$ for any $m \in \boldsymbol{Z}$ and $x \in X$, where f^m for $m = -n$, $n \in \boldsymbol{N}$ is defined $f^m = (f^{-1})^n$ because the inverse bijection f^{-1} to the bijection f exists. It is evident that the above conditions (i), (ii) are satisfied, thus the action $\boldsymbol{A_f}$ is a cascade. Moreover, if (X,f), (Y,g) are mono-unary algebras and $\boldsymbol{A_f} = (\boldsymbol{Z}, X, \Theta_f)$, $\boldsymbol{A_g} = (\boldsymbol{Z}, Y, \Theta_g)$ are corresponding cascades, then for any homomorphism $h: (X, f) \to (Y, g)$ (i. e. $h(f(x)) = g(h(x))$, $x \in X$) we have that $h: \boldsymbol{A_f} \to \boldsymbol{A}_g$ is a homomorphism of cascades, thus the described construction is functorial.

Orbital decompositions of mono-unary algebras

Let us remind the concept of an orbital decomposition of a mono-unary algebra [9, 11]. Let us suppose (X, f) is a mono-unary algebra where the mapping $f: X \to X$ is not necessary bijective or injective. We define a binary relation $\sim_f \subset X \times X$ in this way: For $x, y \in X$ we put $x \sim_f y$ whenever there exists a pair $[m, n] \in \mathbf{N}_0 \times \mathbf{N}_0$ such that $f^n(x) = f^m(y)$. The relation $\sim_f$ is an equivalence on X called a KW-equivalence (Kuratowski-Whyburn). Blocks $S \in X / \sim_f$ are called f-orbits. A subalgebra $(S, f/S)$ (here symbols f/S mean the restriction of the function f onto the set S) of the mono-unary algebra (X, f) is said to be a component of the algebra (X, f) (of course, there holds $f(S) \subset S$). If $\{(S_\alpha, f_\alpha); \alpha \in I\}$ is the system of all components of the algebra (X, f) (here $f_\alpha = f / S_\alpha$) , we write

$$(X, f) = \sum_{\alpha \in I} (S_\alpha, f_\alpha)$$

and this sum is termed as an orbital decomposition of the mono-unary algebra (X, f).

Now let us consider three one-parametrical systems of elementary functions:

$$\varphi_k(x) = x^{2k+1},\ \psi_k(x) = (k+1)x,\ \xi_k(x) = x + k,\ x \in \mathbf{R},\ k \in \mathbf{N}.$$

All considered functions are bijections of the set $\mathbf{R}$ of all real numbers onto itself; functions φ_k have three fixed points $-1, 0, 1$ for any $k \in \mathbf{N}$, functions ψ_k have exactly one fixed point 0, functions ξ_k do not have any fixed point. Denoting by $(0, 1)$ the open interval $\{x \in \mathbf{R}; 0 < x < 1\}$, we obtain the following assertion describing orbital decompositions of mono-unary algebras $(\mathbf{R}, \varphi_k)$, $(\mathbf{R}, \psi_k)$, $(\mathbf{R}, \xi_k)$.

Lemma: The mono-unary algebras $(\mathbf{R}, \varphi_k)$, $(\mathbf{R}, \psi_k)$, $(\mathbf{R}, \xi_k)$ for $k \in \mathbf{N}$ have these orbital decompositions:

$$(\mathbf{R}, \varphi_k) = (\{-1\}, id) + (\{0\}, id) + (\{1\}, id) + \sum_{\alpha \in (0,1)} (X_\alpha, \varphi_{\alpha,k}),$$

$$(\mathbf{R}, \psi_k) = (\{0\}, id) + \sum_{\alpha \in (0,1)} (Y_\alpha, \psi_{\alpha,k}), \quad (\mathbf{R}, \xi_k) = \sum_{\alpha \in (0,1)} (K_\alpha, \xi_{\alpha,k}),$$

where $\varphi_{\alpha,k} = \varphi_k / X_\alpha$, $\psi_{\alpha,k} = \psi_k / Y_\alpha$, $\xi_{\alpha,k} = \xi_k / K_\alpha$, $\alpha \in (0, 1)$ and

$$(X_\alpha, \varphi_{\alpha,k}) \cong (Y_\alpha, \psi_{\alpha,k}) \cong (K_\alpha, \xi_{\alpha,k}) \cong (\mathbf{Z}, \nu_{\mathbf{Z}})$$

for any index $\alpha \in (0, 1)$ and $\nu_{\mathbf{Z}}(m) = m + 1$, $m \in \mathbf{Z}$. □

If (X, f), (Y, g) are mono-unary algebras, let us denote by $Hom((X, f), (Y, g))$ the set of all homomorphisms of the algebra (X, f) into the algebra (Y, g), and similarly, if $\mathbf{A}$, $\mathbf{B}$ are cascades, then $Hom(\mathbf{A}, \mathbf{B})$ means the set of all homomorphisms of the cascade $\mathbf{A}$ into the cascade $\mathbf{B}$.

Constructions of homomorphisms of mono-unary algebras are described in [9], where various modifications and applications of presented constructions are also included. For example, all homomorphisms of the algebra $(\mathbf{R}, \varphi_k)$ into the algebra $(\mathbf{R}, \psi_k)$ can be obtained in this way: Let T be the set of all functions $\tau : (0, 1) \to \langle 0, 1)$ and Λ_α be the set of all isomorphisms

$$\lambda : (X_\alpha, \varphi_{\alpha,k}) \to (Y_{\tau(\alpha)}, \psi_{\tau(\alpha),k})$$

if $\tau(\alpha) \neq 0$ including the constant mapping $\lambda : X_\alpha \to \{0\}$ for $\alpha \in (0, 1)$. Now we put $f(-1) = f(0) = f(1) = 0$, and if $x \in X_\alpha$, we define $f(x) = \lambda(x)$ for a concrete function $\lambda \in \Lambda_\alpha$ and $\tau \in T$. Then $f \in Hom((\boldsymbol{R}, \varphi_k), (\boldsymbol{R}, \psi_k))$ and if the function τ is running over the set T and λ is running over the set Λ_α, $\alpha \in (0, 1)$, we obtain all functions $f \in Hom((\boldsymbol{R}, \varphi_k), (\boldsymbol{R}, \psi_k))$ – [9].

Main results

Let us denote by $\boldsymbol{A}(\varphi_k)$, $\boldsymbol{A}(\psi_k)$, $\boldsymbol{A}(\xi_k)$ cascades assigned by the above presented functorial construction to mono-unary algebras $(\boldsymbol{R}, \varphi_k)$, $(\boldsymbol{R}, \psi_k)$, $(\boldsymbol{R}, \xi_k)$ in the given order. Considering the fact that $card\ T = c^c$, where $c = exp\ \aleph_0$, and that for any function $h : \boldsymbol{R} \to \boldsymbol{R}$ there holds

$$\xi_k(h(0)) = h(0) + k \neq h(0) = h(\varphi_k(0)),\ k \in \boldsymbol{N},$$

and similarly

$$\xi_k(h(0)) = h(0) + k \neq h(0) = h(\psi_k(0)),\ k \in \boldsymbol{N}$$

we obtain with respect to the above lemma and to the mentioned construction of sets of homomorphisms of mono-unary algebras in question the following assertion.

Proposition: For any $k \in \boldsymbol{N}$ we have

$$card\ Hom(\boldsymbol{A}(\varphi_k), \boldsymbol{A}(\psi_k)) = card\ Hom(\boldsymbol{A}(\psi_k), \boldsymbol{A}(\varphi_k)) =$$
$$card\ Hom(\boldsymbol{A}(\xi_k), \boldsymbol{A}(\varphi_k)) = card\ Hom(\boldsymbol{A}(\xi_k), \boldsymbol{A}(\psi_k)) = c^c,$$
$$Hom(\boldsymbol{A}(\varphi_k), \boldsymbol{A}(\xi_k)) = Hom(\boldsymbol{A}(\psi_k), \boldsymbol{A}(\xi_k)) = \phi. \qquad \square$$

The asymmetry expressed in the Proposition can be deleted by the below described construction of extensions.

Definition: Let $\boldsymbol{A}$, $\boldsymbol{B}$ be different cascades such that there exists an injective homomorphism (i.e. an embedding) $h: \boldsymbol{A} \to \boldsymbol{B}$. Then the cascade $\boldsymbol{B}$ is said to be an extension of the cascade $\boldsymbol{A}$.

Theorem: Let us suppose $k \in \boldsymbol{N}$. There exist extensions $\tilde{\boldsymbol{A}}(\tilde{\psi}_k)$, $\hat{\boldsymbol{A}}(\hat{\xi}_k)$ of the cascades $\boldsymbol{A}(\psi_k)$, $\boldsymbol{A}(\xi_k)$ such that

$$\boldsymbol{A}(\varphi_k) \cong \tilde{\boldsymbol{A}}(\tilde{\psi}_k) \cong \hat{\boldsymbol{A}}(\hat{\xi}_k).$$

Proof: The extension of the cascade $\boldsymbol{A}(\psi_k)$ consists in an adding of two different critical points (called also equilibrium points) to its phase set $\boldsymbol{R}$ and also in the extension of the action Θ_ψ onto the new set of three critical points.

Let us denote $\tilde{\boldsymbol{R}} = \boldsymbol{R} \cup \{-\infty, \infty\}$, where $-\infty$, ∞ are elements which do not belong to the set $\boldsymbol{R}$. In terms of operations on the class of ordered sets the chain $(\tilde{\boldsymbol{R}}, \leq)$ can be expressed as the ordinal sum

$$(\tilde{\boldsymbol{R}}, \leq) = \{-\infty\} \oplus (\boldsymbol{R}, \leq) \oplus \{\infty\}.$$

Since

$$\lim_{x\to-\infty} \psi_k(x) = \lim_{x\to-\infty} (k+1)x = -\infty \text{ for any } k \in \boldsymbol{N}$$

and

$$\lim_{x\to\infty} \psi_k(x) = \lim_{x\to\infty} (k+1)x = \infty \text{ for any } k \in \boldsymbol{N}$$

we define $\tilde{\psi}_k(-\infty) = -\infty$, $\tilde{\psi}_k(\infty) = \infty$ and $\tilde{\psi}_k(x) = \psi_k(x)$ for each $x \in \boldsymbol{R}$. Now we have

$$(\tilde{\boldsymbol{R}}, \tilde{\psi}_k) = (\{-\infty\}, id) + (\{0\}, id) + (\infty\}, id) + \sum_{\alpha \in (0,1)} (X_\alpha, \psi_{\alpha,k}),$$

Where – as above $-(X_\alpha, \psi_{\alpha,k}) \cong (\boldsymbol{Z}, \nu_{\boldsymbol{Z}})$ for any $\alpha \in (0, 1)$, thus $(\tilde{\boldsymbol{R}}, \tilde{\psi}_k) \cong (\boldsymbol{R}, \varphi)$, $k \in \boldsymbol{N}$. Then denoting

$$\tilde{\boldsymbol{A}}(\tilde{\psi}_k) = (\tilde{\boldsymbol{R}}, \boldsymbol{Z}, \tilde{\Theta}_{\tilde{\psi}}),$$

where $\tilde{\Theta}_{\tilde{\psi}}(x, m) = (\tilde{\psi})^m(x)$, $x \in \tilde{\boldsymbol{R}}$ and $m \in \boldsymbol{Z}$, we have $\tilde{\boldsymbol{A}}(\tilde{\psi}_k) \cong \boldsymbol{A}(\varphi_k)$ for any $k \in \boldsymbol{N}$. Since for $h(x) = x$, $x \in \boldsymbol{R}$

$$h(\Theta_\psi(x, m)) = \Theta_\psi(x, m) = \Theta_\psi(h(x), m) = \tilde{\Theta}_{\tilde{\psi}}(h(x), m)$$

for each pair $[x, m] \in \boldsymbol{R} \times \boldsymbol{Z}$, the cascade $\tilde{\boldsymbol{A}}(\tilde{\psi}_k)$ is an extension of the cascade $\boldsymbol{A}(\psi_k)$ (here the injection $h: \boldsymbol{R} \to \tilde{\boldsymbol{R}}$ is the corresponding embedding of $\boldsymbol{A}(\psi_k)$ into $\tilde{\boldsymbol{A}}(\tilde{\psi}_k)$ for each $k \in \boldsymbol{N}$.

Mono-unary algebras $(\boldsymbol{R}, \xi_k)$, $k \in \boldsymbol{N}$ do not possess any critical point, hence we add three critical points, namely $x_1 = -\infty$, $x_2 = \infty$ and $x_3 = C_{0,k}$, where $C_{0,k}$ for a given k is the class of the decomposition $\boldsymbol{Z}$ modulo k which forms zero of the group $\boldsymbol{Z}$ / *mod k*, i.e.

$$C_{0,k} = \{\dots, -3k, -2k, -k, 0, k, 2k, 3k, \dots, nk, \dots\}.$$

Similarly as above

$$\lim_{x \to -\infty} \xi_k(x) = -\infty, \ \lim_{x \to \infty} \xi_k(x) = \infty, \ k \in \boldsymbol{N}$$

and moreover

$$\xi_k(C_{0,k}) = C_{0,k} + k = \{z + k;\ z \in C_{0,k}\}$$

for an arbitrary $k \in \boldsymbol{N}$. We define

$$\hat{\boldsymbol{R}}_{\boldsymbol{k}} = \boldsymbol{R} \cup \{-\infty, \infty, C_{0,k}\}, \text{ i.e. } (\hat{\boldsymbol{R}}_{\boldsymbol{k}}, \leq) = \{C_{0,k}\} + (\tilde{\boldsymbol{R}}, \leq)$$

and $\hat{\xi}_k(-\infty) = -\infty$, $\hat{\xi}_k(\infty) = \infty$, $\hat{\xi}_k(C_{0,k}) = C_{0,k}$, $\hat{\xi}_k(x) = \xi_k(x) = x + k$ for any $x \in \boldsymbol{R}$ and $k \in \boldsymbol{N}$.

Then evidently

$$(\boldsymbol{R}, \varphi_k) \cong (\tilde{\boldsymbol{R}}, \tilde{\psi}_k) \cong (\hat{\boldsymbol{R}}_{\boldsymbol{k}}, \hat{\xi}_k) \tag{1}$$

for any $k \in \boldsymbol{N}$. Defining functions $\hat{\Theta}_{k,\hat{\xi}}: \hat{\boldsymbol{R}}_{\boldsymbol{k}} \times \boldsymbol{Z} \to \hat{\boldsymbol{R}}_{\boldsymbol{k}}$ by the formula $\hat{\Theta}_{k,\hat{\xi}}(x, m) = (\hat{\xi}_k)^m(x)$ for any $x \in \hat{\boldsymbol{R}}_{\boldsymbol{k}}$ and any $m \in \boldsymbol{Z}$, we obtain, similarly as above, that cascades $\hat{\boldsymbol{A}}_{\boldsymbol{k}} = (\hat{\boldsymbol{R}}_{\boldsymbol{k}}, \boldsymbol{Z}, \hat{\Theta}_{k,\hat{\xi}})$ are extensions of the cascades $\boldsymbol{A}(\xi_k)$ for any $k \in \boldsymbol{N}$. Consequently, with respect to (1) we have for each $k \in \boldsymbol{N}$

$$\boldsymbol{A}(\varphi_k) \cong \tilde{\boldsymbol{A}}(\tilde{\psi}_k) \cong \hat{\boldsymbol{A}}(\hat{\xi}_k). \qquad \square$$

Concluding remarks

Remark 1: From the just proved theorem there follows immediately that all functional equations

$$f(\rho(x)) = \sigma f(x), \text{ for } \rho, \sigma \in \{\varphi_k, \tilde{\psi}_k, \hat{\xi}_k\}$$

have solution sets of the cardinality c^c and moreover all of the mentioned functional equations have infinitely many bijective solutions.

Remark 2: Treated structures are very special objects belonging to the theory of dynamical systems or to dynamical topology. Bibliography for these topics up to 1972 was compiled by Walter Helbig Gottschalk, Wesleyan University, Middletown, Connecticut. This Bibliography contains 178 pages and lists 2475 numbered items – cf. [7], p. 164 or [4], p. 301. Further, discrete dynamical systems with input discrete groups or semigroups constitute in fact a certain class of automata without outputs also called quasi-automata – [5, 8]. Literature devoted to these structures belonging into the algebraic theory of automata is very comprehensive. Let us notice in this connection that the actions of groups from the purely algebraic point of view are treated in chapter 5 of the monography [6] and in the paper [8].

Remark 3: In the theory of dynamical systems, in particular in the theory of continuous flows – [2] there are investigated some concepts and properties of objects which can be transferred onto cascades. Chapter II, [2] of the monography [2] contains definitions of invariant, positively invariant and negatively invariant subsets of a phase set of a flow. The mentioned notions can be transferred onto cascades without any formal change. Also theorems 1. 3. through 1. 5., pp. 12-13 [2] are valid. In particular, theorem 1. 5. says that a set $M \subset X$ (the phase set) is invariant (i. e. $\Theta(x, p) \in M$ for all $x \in M$ and all $p \in \boldsymbol{Z}$ in our case) if and only if it is both positively invariant ($\Theta(x, p) \in M$ for all $x \in M$ and all $p \in \boldsymbol{Z}^+$) and negatively invariant (similarly as above, but $p \in \boldsymbol{Z}^-$). Since a set $M \subset \boldsymbol{R}$ is invariant in a cascade $\boldsymbol{A} = (\boldsymbol{R}, \boldsymbol{Z}, \Theta)$ if and only if M is the union of a system of orbits, it is clear that theorem 1. 5. [2] is also true in our case. Also the concept of a trajectory $\gamma(x)$ [2] (or a semi-trajectory) in our case is a single orbit (or an iterated sequence – i. e. a splinter [10]) containing the number $x \in \boldsymbol{R}$. Terms of the first positive prolongation and of the first prolongational limit set are near to our construction of extensions of cascades. Thus, all the mentioned concepts can be illustrated by our considered structures.

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