Students' Creativity in Problem Solving

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Abstract

This article reports results from a research project carried out in upper secondary school in collaboration with eight Norwegian mathematics teachers. The project concentrated on the development of students' inquiry, creativity and intellectual independence while they were working in a problem solving setting in mathematics classes. The teachers prepared and conducted teaching experiments for the inquiry of the students' strategies. Two episodes of creativity will be presented and discussed in thearticle based on excerpts from the data. Works of Schoenfeld, Polya, Johan Lithner and A.A. diSessa form the theoretical basis for the project.

Keywords: Investigation and problem solving, meta representational competence, mathematical creativity, mathematically founded reasoning.

Classification: C30, D40, D50, D20

Introduction

This project, running with the aim to develop and study teaching that encourages students' activity, inquiry and autonomy, is part of the EU project KeyCoMath (http://www.keycomath.eu/). It started in the spring 2013 where a collaborative research group was formed consisting of eight mathematics teachers from five local, upper secondary schools and one university researcher in mathematics education (Me, the author of this article). The group had articulated certain concerns like: i) Students are too dependent of check lists and working habits; they seldom are able to 'think outside the box' ii) Even the brightest students can reproduce, but rarely produce mathematical thinking and iii) Many students do not want to solve new problems or to answer new questions.

The group had the hypothesis, that appropriate problem-solving environments could support realization of many students' hidden potentials for independent, mathematical thinking. This hypothesis was founded on the teachers' experiences and on reading in the research group of (Schoenfeld 2011) and others. The teachers already knew (Polya 1985) and wanted to use Polya's scheeme for solving mathematical problems.

We formulated the research question: "What strategies can we identify when the students work in an inquiry based learning environment in upper secondary mathematics?" During the first year of the project, the teachers had designed and taught sequences in their own classes of about ten lessons each, where the students worked with problem solving. Gradually, our group's research interest concentrated on students' modes of reasoning and, in particular, on ideas about mathematical creativity presented by Lithner (2008). The

teachers deliberately designed sequences, which were meant to provoke examples of mathematical creativity. During the following years, we studied these teaching experiments and analyzed data with the aim to study episodes of mathematical creativity. We designed and taught new teaching experiments based on experiences from the first round. Analysis of these latest data is still going on.

Glimpses of mathematical creativity

One important result of the project was a didactic concept, which emerged during the teaching experiments: a particular aspect of the students' work, which we interpreted as glimpses of mathematical creativity (GMC). In the following, I present two episodes of GMC, picked out from the project's data. The GMCs occurred when pairs of students worked together in the experimental learning environment. Characteristic of the environment was the demand, that the students engaged in solving a mathematical problem which was completely new to them, and also that the teacher deliberately would avoid to interfere by, for example, asking the students sub questions or structuring their actual process of problem solving.

Analysis and discussion of GMC as a didactic concept evolving from the episodes take place in the article's final section, based on the research project's theoretical framework. The main issues for discussion are: "Can we interpret GMC as one type of Creative Mathematically founded Reasoning (CMR) (Lithner 2008)?" and "What are the connections between GMC and Meta Representational Competence (MRC) (DiSessa 2002)?"

Theoretical framework in brief

Lithner: Types of reasoning for solving tasks

According to (Lithner, 2008), solving a task can be seen as carrying out four steps:

1) A (sub) task is met, which is denoted problematic situation if it is not obvious how to proceed.
2) A strategy choice is made. It can be supported by predictive argumentation: Why will the strategy solve the task?
3) The strategy is implemented, which can be supported by verificative argumentation: Why did the strategy solve the task?
4) A conclusion is obtained.

Further, Lithner discerns between different types of reasoning involving strategy choice and strategy implementation. The two main types of reasoning are IR (Imitative Reasoning) and CMR (Creative Mathematically founded Reasoning). IR encompasses i) memorised reasoning where the strategy choice is founded on recalling a complete answer and the strategy implementation consists only of writing it down, and ii) three subtypes of algorithmic reasoning (AR) where the strategy choice is to recall a solution algorithm without creating a new solution; hereafter, the remaining parts of the strategy implementation are trivial.

In contrast, CMR fulfils all of the following criteria (Lithner, 2008) p 266:

1) Novelty. A new (to the reasoned) reasoning sequence is created, or a forgotten one is re-created.
2) Plausibility. There are arguments supporting the strategy choice and/or strategy implementation motivating why the conclusions are true or plausible.

3) Mathematical foundation. The arguments are anchored in intrinsic mathematical properties of the components involved in the reasoning.

Lithner did his studies at undergraduate level. Our group decided to take students' CMR as a goal for the teaching experiment. Therefore, the data analysis concentrated on the identification of episodes of students' creative mathematical thinking.

diSessa: meta representational competence (MRC)

Meta representational competence (MRC) refers to the full complex of abilities to deal with representational issues. It includes, centrally, the ability to design new representations, including both creating representations and judging their adequacy for particular purposes. But it also includes understanding how presentations work, how to work presentations for different purposes and, indeed, what the purposes of representations are. Knowledge that allows students to learn new representations quickly and the ability to explain representations and their properties is also included (diSessa 2002). Representational literacy is important for the students' critical capabilities (meaning the capability of judging the effectiveness of the design's result, and of redesigning it) in MRC, according to (diSessa 2002). According to diSessa (2004), MRC may account for some parts of the competence to learn new concepts and to solve novel problems. Our group's observations were in line with this and gave inspiration to new inquiries. Further, diSessa (2004) suggests that because insight and competence often involve coming up with an appropriate representation, learning may implicate developing one's own personally effective representations for dealing with a conceptual domain.

Although these two studies (diSessa 2002, 2004), in contrast with our project, aim at linking meta representational competence with design, and with students' critical capabilities, we found the concept of meta representational competence potentially useful for analysis of the GMC's occurring from our data.

Two episodes of GMC

The two episodes took place in different parts of the project. Nevertheless, they have something in common: After a period of work and unsuccessful trials, one of the students suddenly see a solution in a glimpse. He or she explains it to the other student(s) who immediately accept the solution. Subsequent data analysis does not reveal a clear connection between the solution and the two students' proceeding ideas or suggestions. The solution may even diverge from the solution and the learning trajectory envisioned by the teacher when he or she designed the teaching experiment.

Episode 1

This episode is the English version of an episode that I have presented and discussed in (Andresen 2015a). The students in the classroom had been introduced to sequences and they had solved tasks from the textbook based on algebraic expressions and numerical examples. As the first part of the experimental teaching sequence, they had solved the task (Figure 1) on the blackboard (Nelsen 1993). The task was to find the connection between the figure and the algebraic expression, and thereby, to use the figure as an argument for the expression. In the episode's task, the students were asked to solve a similar problem (Figure 2) (Nelsen 1993)
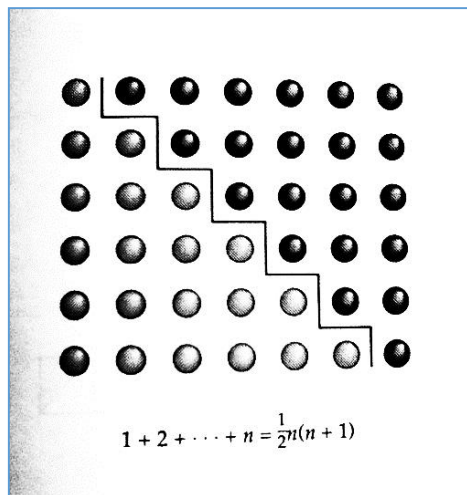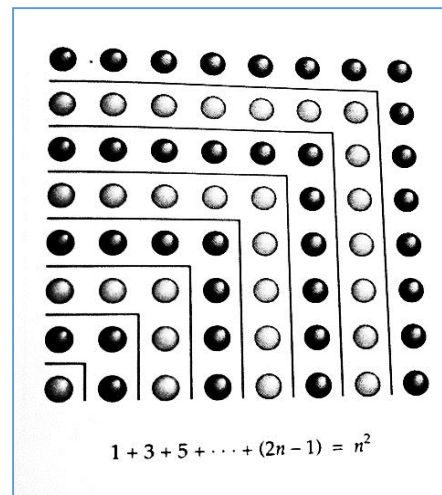
Figure 1

Figure 2

The results of the two students' work are reconstructed here (Figure 3):

First version	First revised version	Corrected version with help from E3
$S_1=1$ $S_2=1+2$ or $1+3$? Writes: $1+2=3$ $S_3=1+2+3=6$ $S_4=1+2+3+4=10$ $S_5=1+2+3+4+5=15$ $S_6=1+2+3+4+5+6=21$ $S_7=1+2+3+4+5+6+7=28$ $S_8=1+2+3+4+5+6+7+8=36$	$S_2=2+1$ $S_3=3+3$, $S_4=4+6$ $S_5=5+10$ $S_6=6+15$ $S_7=7+21$ $S_8=8+28$	$S_1=1$ $S_2=1+3=4$ $S_3=1+3+5=9$ $S_4=1+3+5+7=16$ $S_5=1+3+5+7+9=25$ $S_6=1+3+5+7+9+11=36$ $S_7=1+3+5+7+9+11+13=4$ $S_8=1+3+5+7+9+11+13+15=64$

Figure 3

Figure 4 entails the transcription of the episode (my translation) in the left column and my reflections and explanations in the right column:

Episode 1 (Tanks 02.09.2013 video5)	
E1: (writing,) $S_1=1$, $S_2=1+2$ or $1+3$? E2: $1+3$ but E1 insists on writing $1+2$,	The dialog between E1 and E2 shows how they take the preceeding task as their starting point (Figure 1) and try to copy the strategy from that one. They continue (My reconstruction Figure 3, 1. column). E2 suggests that they write $1+3$, but he is overruled by E1 without any arguments
E2: (points) then we must find out if this, plus something, equals that (the sum) (..) They agree about the expression $S_2=2+1$, and use the corresponding expressions up to S_8.	The students do not discern clearly between a_n and S_n, and see no clear connection to the 'rows' and the 'area', respectively, on the figure. Consequently, they cannot establish a pattern but merely write the expressions from the preceeding task a_i equals the number of elements in diagonal number i. This strategy is similar to Lithner's AR.
They try to find a recursive expression (...) E1: (writing) $a_n=a_{n-1}+2$ E2: Is it true? This is S_n E1: S_2 (points to what she has written)	Hereafter, when they continue looking for a pattern in the expressions for S_1, S_2, ..., they create a way to split the numbers in two terms, which are dependent of i (2. column Figure 3).

equals 2-1+2, equals 3 (writes) $S_n = S_{2-1} + 2 = 3$	
E2: But S_{3-1} , then? E1: And now $S_p = S_{p-1} + 2$ – oups! Something is wrong here! We have found something totally different..	This splitting appears to be of no help since it does not link to the 'area' of the figure. They agree about giving it up
E2: It is just plus in stead of minus, p+1, +1+2, equals 6 Ida: Oh yes!	This looks like guessing and they do not use it
E3 (sitting in front of the two, at the next table) turns around to help them E3: But this is not correct? E1: Yes, E3: How could it be correct? S_7 equals 28? E1: Yes, we added all of them, the result E3: But take a look here (points to the figure) you must add $1 + 3 + ...$ E1: Oh yes, we did only add 1+2+3 (points to the columns) E3: It is boring to write all this (points to the sums), just write S_1 E1: 4, E3: no, this is S_1	With help from another student (E3) S_n is now linked with the 'area' of square number n. E3 refers to the results in 1. column Figure 3. when she says that it is not correct. E1 defends their result by claiming that they have added all the numbers and got a result(!), but she is easily convinced when E3 points to the figure and claims that they added the wrong numbers E3 tries to convince them about the shorter, generalized term S_i by saying that it is boring to write many numbers, maybe she repeat what the teachers used to say
E1: (writing) $S_1 = 1$, $S_2 = 4$, $S_3 =$ (counting)5,6,7,8,9, (writes) 9, $S_4 =$ (counting) 10,11,12,13,etc. E1 continues the counting, writes correctly up to S_8..	E1 finds the values of S_i (3. column Figure 3) by counting the bullets on the figure.
Now they have to see a pattern and again, the try sums E1: And now we must multiply – no add.. E2: S_2, then we must take for example 2.. E1: Plus 2 (writes) 2+2	Apparently, E2 and E1 choose randomly between addition and multiplication when they continue with S_n. At least, they give no arguments
E2: 2 times 4 is not of any help.. E1: no, E2: So we start with 2+2. $S_1 = 1$, $S_2 = 4 = 2+2$, E2 suggests 2 times 2 +1 but they agree about adding: $S_3 = 3+6$	E2 mentions that one of the values of a S_n might appear as 2 times 4 but immediately rejects the idea
E3 (points to the numbers) E1: oh – square numbers? E3: Yes,(writes) $S_n = n^2$, we have already figured it out	They choose the same pattern as before but E3 stops them. E3 points to the new values and in that way, she makes E1 og E2 aware of the square numbers. Here E1 and E2 experience the GMC.

Figure 4

Episode 2

This episode was presented and discussed in (Andresen 2015b). It took place in a highest-level mathematics classroom with about 22 students. The teacher gave an introduction of Polygonal numbers, based on his oral explanation of how the next polygonal number emerged from the previous by expanding the polygon, and based on his drawings on the blackboard (Figure 5).
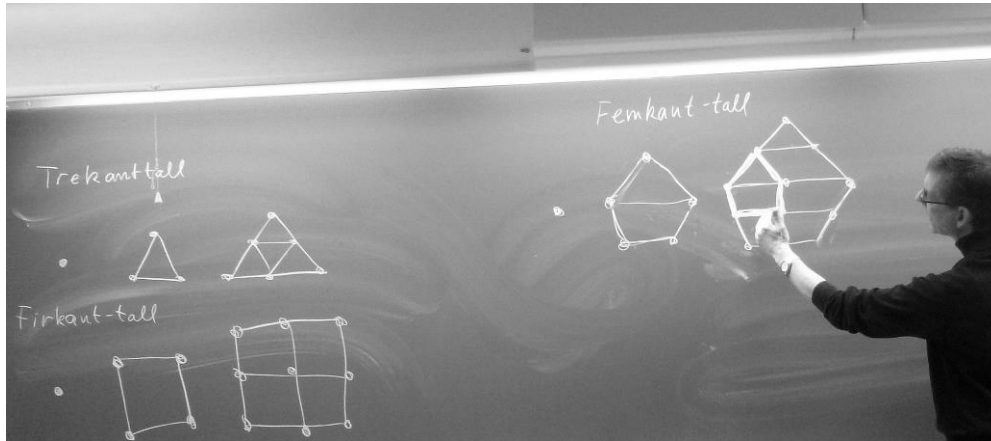

Figure 5

The students' task was to complete a form, distributed by the teacher, with the polygonal numbers and to express the general terms (Figure 6). After the introduction, the students started to work in pairs. The subject polygonal numbers was new to the students and they had no prior experiences (from the classroom, according to the teacher) with this kind of tasks.

Figurtallene (polygontallene)

Pytagoreerne jobbet mye med disse tallene ca 500 f.Kr.

Vi skal lete etter tallmønstre og beskrive dem med matematikk

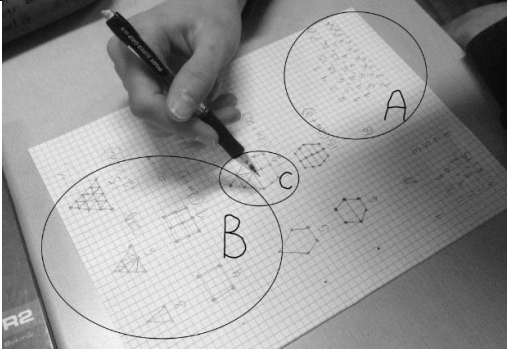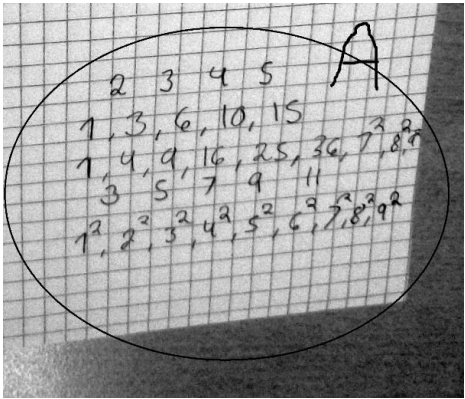| Figurtall	Symbol	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	Generell n
Trekanttall	t_n	1	3	6	10	15	21	
Firkanttall	f_n	1	4	9				
Femkanttall	p_n	1	5					
Sekskanttall	h_n	1	6					
Syvkanttall	s_n	1	7					
Åttekanttall	o_n	1	8					
k-kanttall	k_n	1	k					

Figure 6

There was no restrictions on what methods they might to use, but the teacher gave no hints or sub questions, neither. One strategy for completing the form would be to study the pattern of increase, as shown in Figure 7.
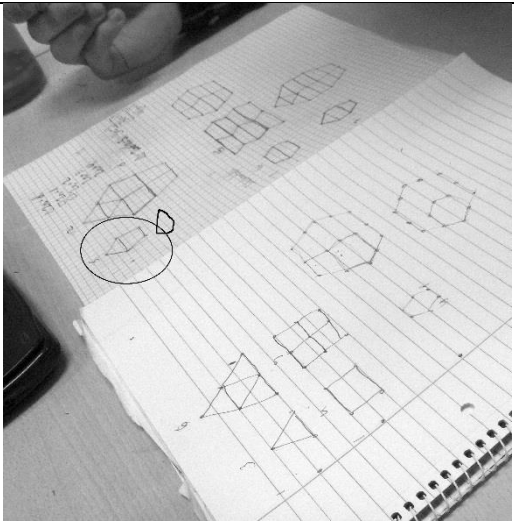
	n=1	n=2	n=3	n=4	n=5	n=6	General
Triangular Numbers	1	3	6	10	15	21	$\frac{1}{2} \cdot n(n+1)$
Increased by	2	3	4	5	6	n	
Square numbers	1	4	9	16	25	36	n^2
Increased by	2·2-1	2·3-1	2·4-1	2·5-1	2·6-1	2·n-1	
Pentagonal numbers	1	5	12	22	35	51	$\frac{3}{2} n^2 - \frac{1}{2}n$
Increased by	3·2-2	3·3-2	3·4-2	3·5-2	3·6-2	3·n-2	
Hexagonal numbers	1	6	15	28	45	66	$2n^2 - n$
Increased by	4·2-3	4·3-3	4·4-3	4·5-3	4·6-3	4·n-3	
k-polygonal numbers	1	k	3(k-2)-(k-3)+k	...	...	...	$\frac{1}{2} k (n^2 - n) - n^2 + 2n$
Increased by		3(k-2)-(k-3)	4(k-2)-(k-3)	5(k-2)-(k-3)	6(k-2)-(k-3)	n(k-2)-(k-3)	

Figure 7

Most of the students combined drawings with counting and, simultaneously, looked for patterns in the rows and/or columns containing the numbers obtained from the drawings.

Figure 8 entails the transcription of the episode (my translation) in the left column and my reflections and explanations in the right column:

	Episode 2 (Danielsen 18.03.2014 video8)
1	The two students B1 and B2 sit and work together. They have already managed to write the first five triangular numbers 1, 3, 6, 10 and 15 and the square numbers 1, 4, 9, 16, 25, 36, 7^2 8^2 and 9^2 (Area A Pictures a and b) based on their drawings (Area B Picture a). Apparently, their unarticulated plan was to find a pattern for the extension from triangular numbers to square numbers, which they could extend to create the pentagular numbers and, afterwards, the succeeding polygonal numbers. B1:..*Then the next one is seven squared,* (writes 7^2),*the next one is eight squared* (writes 8^2), *the next is nine squared* (writes 9^2 in Area A Pictures a and b),*	 *Picture a*

		Picture b
2	B1: *then we know the difference between these* (points to the square numbers, points to the numbers 3, 5, 7, 9, 11 in area A Pictures a and b)	*Their preliminary choice of a strategy was, apparently, to read a pattern from the increase of the square numbers. The teacher's introduction had lead them in this direction (without giving any details, though)*
3	B1: *.. so in fact you have* (writes 1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^2, 9^2, last line in area A, Pictures a and b)	*B1 rewrites the square numbers in powers of two, apparently for making it easier to read a pattern*
4	B2: *But we cannot..* B1: *How can we write a formula for this?* B2: *For the triangle, it is not squared at least* B1: *But the triangle is different (..)*	It becomes clear to B1 and B2 that the pattern they look for cannot be as simple as an increase in powers
5	B1: *The triangle, it is something with its three sides, with the triangle in the middle somehow..* (points to area C, Picture a) B1: (draws a triangle, covered by his hand on Picture a)	30 seconds silence B1 and B2 are both starring at their drawings. According to my interpretation, they reconsider the strategy and try to take inspiration for a new strategy
6	B1:*(..) one more. What is the formula for the square?* B2: *Yes I see that, the square is okay. But..*	This sounds as if B1 still considers the old strategy of extension, and maybe he wants to check it out again. B2 is finished with the squares and he does not reconsider the same extension idea
7	B2: *But then, the triangle, you can somehow..* (points to the polygon in area D on his drawing Picture c)	*Picture c*
8	B2: *For example, for the pentagonal, then you may in a way, you can take a formula for the triangle and a formula*	B2 takes inspiration from his drawing to express the pentagonal numbers by a formula, which he can create by adding the

	for the square and add them	formulas they already know.
9	B1: *Then you can use it for all* B2: *Yes you can do it with all of them* B1: *Yes, exactly. So This is the square* (points to the squares in area B in Picture a), *that is why it becomes like this ..*	B1 acknowledges that the principle is applicable for all the pentagonal numbers and B2 agrees. B1 recognises the squares as parts of the pentagonal numbers in the first few cases in his own drawings
10	B1: *For example these points here, they have two in common..* B2: *Yes yes..*	They start to figure the formula out as a sum when taking into account that the triangle and the square has one line in common, according to the drawing

Figure 8

Discussion and Conclusion

The discussion in (Andresen 2015a) of episode 1 concludes that the episode contains an example of the algorithmic reasoning (AR) described by Lithner (2008), first Column Figure 3, where the students copy the solution of the preceeding task. It also contains examples of initial creative thinking in the second and third column, Figure 3. All the episode's examples of creative thinking concern elementary reasoning rather than problem solving but they took place in a problem solving setting.

In parallel, the discussion in (Andresen 2015b) of episode 2 compares the episode with Lithner's criteria for CMR (Lithner 2008): The four criteria for CMR were fulfilled in the case, only if shifts between different representations can be seen as part of the 'mathematical foundation' in the third one. The GMC in episode 2 was founded on the students' competence in shifting between the different representations (numbers, formulas and drawings) of the polygonal numbers. Their arguments for supporting the strategy choice were anchored in both students' representational literacy, which is an aspect of meta representational competence (MRC) as it was described above.

One of the students' representational literacy was revealed in the episode's scheme row 8, where B2 talked about the formula for the triangle and for the square, and about adding these two, without even to discern between the different representations. The other student immediately understood the idea. Episode 2 illustrates how the experimental lesson on polygonal numbers, founded on interplay between different representations, could be supportive of the students' development of meta representational competence as well as their creative, mathematically founded reasoning.

The student's creative reasoning in both episodes happened in a glimpse. In episode 1, it happened when the students E1 and E2 realised that they had squares and square numbers. In episode 2, it happened when the student B2 caught the connection between the pentagon consisting of a triangle (with a corresponding formula) and a square (with a corresponding formula) on the one hand, and, on the other hand, the algebraic number of which he wanted to have a formula.

Hence, in both episodes the GMC happened in the moment, when the student managed to see the two as different representations of the same object. A number of episodes from our data contains examples of GMC, which happen in a similar way when a student manage to establish a link between two different representations of the same object. Further analysis of data from our group's experiments may provide interesting insight into the connections and relations between CMR, GMC and MRC.

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