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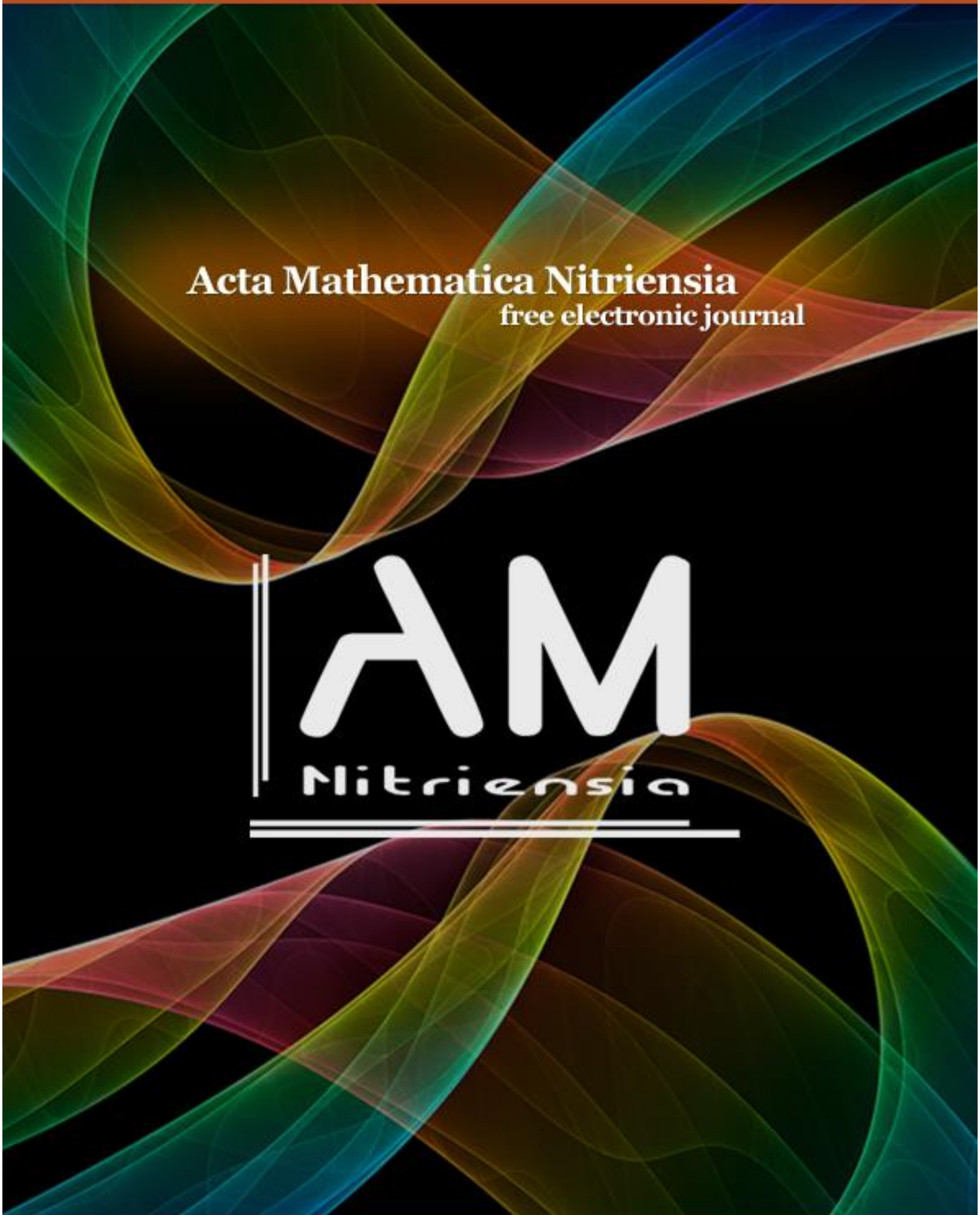

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Univerzita Konštantína Filozofa v Nitre
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Students' Creativity in Problem Solving

Mette Andresen*

Department of Mathematics, University of Bergen, Postbox 7803, N - 5020 Bergen, Norway

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Abstract

This article reports results from a research project carried out in upper secondary school in collaboration with eight Norwegian mathematics teachers. The project concentrated on the development of students' inquiry, creativity and intellectual independence while they were working in a problem solving setting in mathematics classes. The teachers prepared and conducted teaching experiments for the inquiry of the students' strategies. Two episodes of creativity will be presented and discussed in thearticle based on excerpts from the data. Works of Schoenfeld, Polya, Johan Lithner and A.A. diSessa form the theoretical basis for the project.

Keywords: Investigation and problem solving, meta representational competence, mathematical creativity, mathematically founded reasoning.

Classification: C30, D40, D50, D20

Introduction

This project, running with the aim to develop and study teaching that encourages students' activity, inquiry and autonomy, is part of the EU project KeyCoMath (http://www.keycomath.eu/). It started in the spring 2013 where a collaborative research group was formed consisting of eight mathematics teachers from five local, upper secondary schools and one university researcher in mathematics education (Me, the author of this article). The group had articulated certain concerns like: i) Students are too dependent of check lists and working habits; they seldom are able to 'think outside the box' ii) Even the brightest students can reproduce, but rarely produce mathematical thinking and iii) Many students do not want to solve new problems or to answer new questions.

The group had the hypothesis, that appropriate problem-solving environments could support realization of many students' hidden potentials for independent, mathematical thinking. This hypothesis was founded on the teachers' experiences and on reading in the research group of (Schoenfeld 2011) and others. The teachers already knew (Polya 1985) and wanted to use Polya's scheeme for solving mathematical problems.

We formulated the research question: "What strategies can we identify when the students work in an inquiry based learning environment in upper secondary mathematics?" During the first year of the project, the teachers had designed and taught sequences in their own classes of about ten lessons each, where the students worked with problem solving. Gradually, our group's research interest concentrated on students' modes of reasoning and, in particular, on ideas about mathematical creativity presented by Lithner (2008). The

*Corresponding author: Mette.Andresen@uib.no
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teachers deliberately designed sequences, which were meant to provoke examples of mathematical creativity. During the following years, we studied these teaching experiments and analyzed data with the aim to study episodes of mathematical creativity. We designed and taught new teaching experiments based on experiences from the first round. Analysis of these latest data is still going on.

Glimpses of mathematical creativity

One important result of the project was a didactic concept, which emerged during the teaching experiments: a particular aspect of the students' work, which we interpreted as glimpses of mathematical creativity (GMC). In the following, I present two episodes of GMC, picked out from the project's data. The GMCs occurred when pairs of students worked together in the experimental learning environment. Characteristic of the environment was the demand, that the students engaged in solving a mathematical problem which was completely new to them, and also that the teacher deliberately would avoid to interfere by, for example, asking the students sub questions or structuring their actual process of problem solving.

Analysis and discussion of GMC as a didactic concept evolving from the episodes take place in the article's final section, based on the research project's theoretical framework. The main issues for discussion are: "Can we interpret GMC as one type of Creative Mathematically founded Reasoning (CMR) (Lithner 2008)?" and "What are the connections between GMC and Meta Representational Competence (MRC) (DiSessa 2002)?"

Theoretical framework in brief

Lithner: Types of reasoning for solving tasks

According to (Lithner, 2008), solving a task can be seen as carrying out four steps:

1) A (sub) task is met, which is denoted problematic situation if it is not obvious how to proceed.
2) A strategy choice is made. It can be supported by predictive argumentation: Why will the strategy solve the task?
3) The strategy is implemented, which can be supported by verificative argumentation: Why did the strategy solve the task?
4) A conclusion is obtained.

Further, Lithner discerns between different types of reasoning involving strategy choice and strategy implementation. The two main types of reasoning are IR (Imitative Reasoning) and CMR (Creative Mathematically founded Reasoning). IR encompasses i) memorised reasoning where the strategy choice is founded on recalling a complete answer and the strategy implementation consists only of writing it down, and ii) three subtypes of algorithmic reasoning (AR) where the strategy choice is to recall a solution algorithm without creating a new solution; hereafter, the remaining parts of the strategy implementation are trivial.

In contrast, CMR fulfils all of the following criteria (Lithner, 2008) p 266:

1) Novelty. A new (to the reasoned) reasoning sequence is created, or a forgotten one is re-created.
2) Plausibility. There are arguments supporting the strategy choice and/or strategy implementation motivating why the conclusions are true or plausible.

3) Mathematical foundation. The arguments are anchored in intrinsic mathematical properties of the components involved in the reasoning.

Lithner did his studies at undergraduate level. Our group decided to take students' CMR as a goal for the teaching experiment. Therefore, the data analysis concentrated on the identification of episodes of students' creative mathematical thinking.

diSessa: meta representational competence (MRC)

Meta representational competence (MRC) refers to the full complex of abilities to deal with representational issues. It includes, centrally, the ability to design new representations, including both creating representations and judging their adequacy for particular purposes. But it also includes understanding how presentations work, how to work presentations for different purposes and, indeed, what the purposes of representations are. Knowledge that allows students to learn new representations quickly and the ability to explain representations and their properties is also included (diSessa 2002). Representational literacy is important for the students' critical capabilities (meaning the capability of judging the effectiveness of the design's result, and of redesigning it) in MRC, according to (diSessa 2002). According to diSessa (2004), MRC may account for some parts of the competence to learn new concepts and to solve novel problems. Our group's observations were in line with this and gave inspiration to new inquiries. Further, diSessa (2004) suggests that because insight and competence often involve coming up with an appropriate representation, learning may implicate developing one's own personally effective representations for dealing with a conceptual domain.

Although these two studies (diSessa 2002, 2004), in contrast with our project, aim at linking meta representational competence with design, and with students' critical capabilities, we found the concept of meta representational competence potentially useful for analysis of the GMC's occurring from our data.

Two episodes of GMC

The two episodes took place in different parts of the project. Nevertheless, they have something in common: After a period of work and unsuccessful trials, one of the students suddenly see a solution in a glimpse. He or she explains it to the other student(s) who immediately accept the solution. Subsequent data analysis does not reveal a clear connection between the solution and the two students' proceeding ideas or suggestions. The solution may even diverge from the solution and the learning trajectory envisioned by the teacher when he or she designed the teaching experiment.

Episode 1

This episode is the English version of an episode that I have presented and discussed in (Andresen 2015a). The students in the classroom had been introduced to sequences and they had solved tasks from the textbook based on algebraic expressions and numerical examples. As the first part of the experimental teaching sequence, they had solved the task (Figure 1) on the blackboard (Nelsen 1993). The task was to find the connection between the figure and the algebraic expression, and thereby, to use the figure as an argument for the expression. In the episode's task, the students were asked to solve a similar problem (Figure 2) (Nelsen 1993)

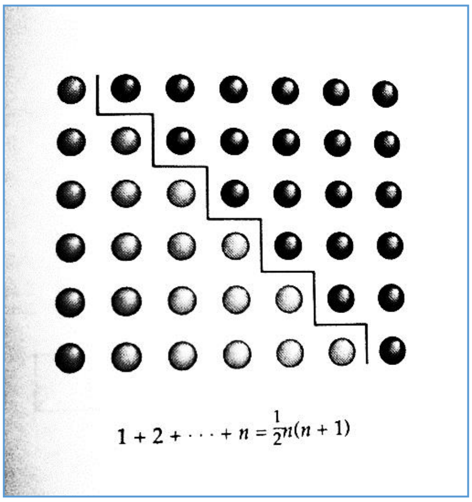


Figure 1

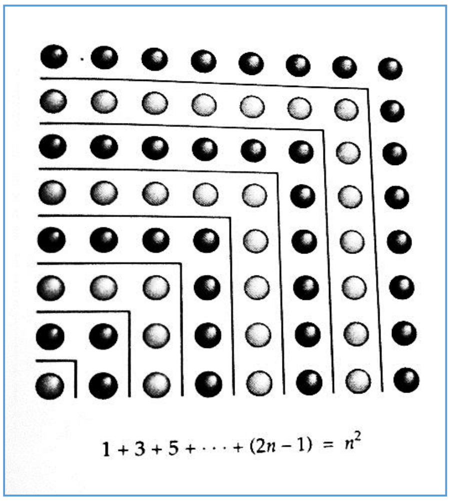


Figure 2

The results of the two students' work are reconstructed here (Figure 3):

First version	First revised version	Corrected version with help from E3
S_1=1 S_2=1+2 or 1+3? Writes: 1+2=3 S_3=1+2+3=6 S_4=1+2+3+4=10 S_5=1+2+3+4+5=15 S_6=1+2+3+4+5+6=21 S_7=1+2+3+4+5+6+7=28 S_8=1+2+3+4+5+6+7+8=36	 S_2=2+1 S_3=3+ 3, S_4=4+6 S_5=5+10 S_6=6+15 S_7=7+21 S_8=8+28	S_1=1 S_2=1+3=4 S_3=1+3+5=9 S_4=1+3+5+7=16 S_5=1+3+5+7+9=25 S_6=1+3+5+7+9+11=36 S_7=1+3+5+7+9+11+13=4 S_8=1+3+5+7+9+11+13+15=64

Figure 3

Figure 4 entails the transcription of the episode (my translation) in the left column and my reflections and explanations in the right column:

Episode 1 (Tanks 02.09.2013 video5)	
E1: (writing,) S_1=1, S_2=1+2 or 1+3? E2: 1+3 but E1 insists on writing 1+2,	The dialog between E1 and E2 shows how they take the preceeding task as their starting point (Figure 1) and try to copy the strategy from that one. They continue (My reconstruction Figure 3, 1. column). E2 suggests that they write 1+3, but he is overruled by E1 without any arguments
E2: (points) then we must find out if this, plus something, equals that (the sum) (..) They agree about the expression S_2=2+1, and use the corresponding expressions up to S_8.	The students do not discern clearly between a_n and S_n , and see no clear connection to the 'rows' and the 'area', respectively, on the figure. Consequently, they cannot establish a pattern but merely write the expressions from the preceeding task a_i equals the number of elements in diagonal number i. This strategy is similar to Lithner's AR.
They try to find a recursive expression (...) E1: (writing) a_n=a_{n-1}+2 E2: Is it true? This is S_n E1: S_2 (points to what she has written)	Hereafter, when they continue looking for a pattern in the expressions for S_1, S_2, ..., they create a way to split the numbers in two terms, which are dependent of i (2. column Figure 3).

equals 2-1+2, equals 3 (writes) $S_n=S_{2-1}+2=3$	
E2: But S_{3-1} , then? E1: And now $S_p=S_{p-1}+2$ – oups! Something is wrong here! We have found something totally different..	This splitting appears to be of no help since it does not link to the 'area' of the figure. They agree about giving it up
E2: It is just plus in stead of minus, p+1, +1+2, equals 6 Ida: Oh yes!	This looks like guessing and they do not use it
E3 (sitting in front of the two, at the next table) turns around to help them E3: But this is not correct? E1: Yes, E3: How could it be correct? S_7 equals 28? E1: Yes, we added all of them, the result E3: But take a look here (points to the figure) you must add 1 + 3 + ... E1: Oh yes, we did only add 1+2+3 (points to the columns) E3: It is boring to write all this (points to the sums), just write S_1 E1: 4, E3: no, this is S_1	With help from another student (E3) S_n is now linked with the 'area' of square number n. E3 refers to the results in 1. column Figure 3. when she says that it is not correct. E1 defends their result by claiming that they have added all the numbers and got a result(!), but she is easily convinced when E3 points to the figure and claims that they added the wrong numbers E3 tries to convince them about the shorter, generalized term S_i by saying that it is boring to write many numbers, maybe she repeat what the teachers used to say
E1: (writing) $S_1=1$, $S_2=4$, $S_3=$(counting)5,6,7,8,9, (writes) 9, $S_4=$(counting) 10,11,12,13,etc. E1 continues the counting, writes correctly up to S_8..	E1 finds the values of S_i (3. column Figure 3) by counting the bullets on the figure.
Now they have to see a pattern and again, the try sums E1: And now we must multiply – no add.. E2: S_2, then we must take for example 2.. E1: Plus 2 (writes) 2+2	Apparently, E2 and E1 choose randomly between addition and multiplication when they continue with S_n. At least, they give no arguments
E2: 2 times 4 is not of any help.. E1: no, E2: So we start with 2+2. $S_1=1$, $S_2=4=2+2$, E2 suggests 2 times 2 +1 but they agree about adding: $S_3=3+6$	E2 mentions that one of the values of a S_n might appear as 2 times 4 but immediately rejects the idea
E3 (points to the numbers) E1: oh – square numbers? E3: Yes,(writes) $S_n=n^2$, we have already figured it out	They choose the same pattern as before but E3 stops them. E3 points to the new values and in that way, she makes E1 og E2 aware of the square numbers. Here E1 and E2 experience the GMC.

Figure 4

Episode 2

This episode was presented and discussed in (Andresen 2015b). It took place in a highest-level mathematics classroom with about 22 students. The teacher gave an introduction of Polygonal numbers, based on his oral explanation of how the next polygonal number emerged from the previous by expanding the polygon, and based on his drawings on the blackboard (Figure 5).

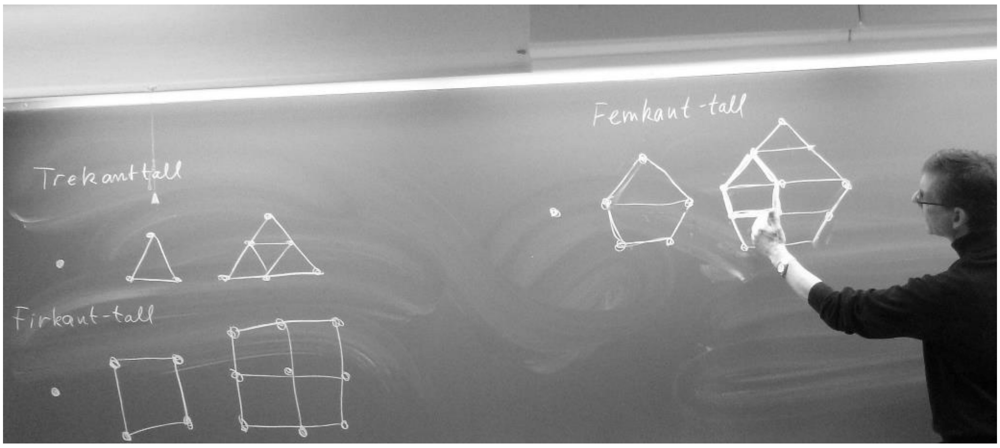
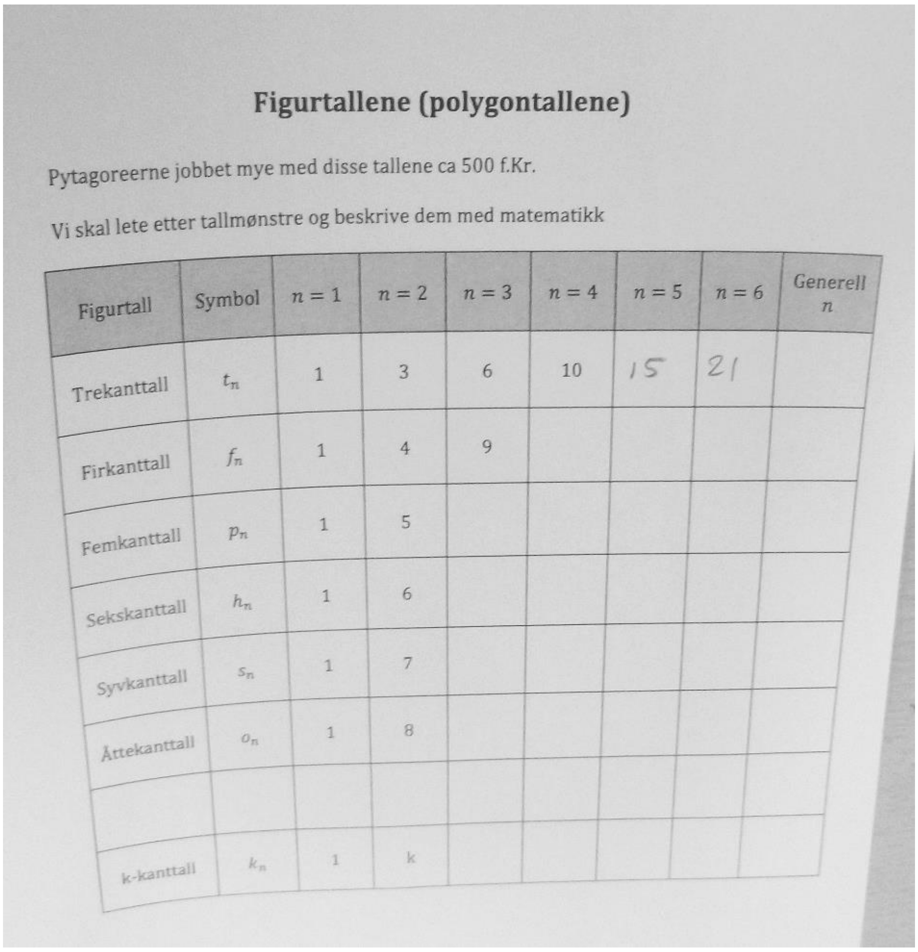


Figure 5

The students' task was to complete a form, distributed by the teacher, with the polygonal numbers and to express the general terms (Figure 6). After the introduction, the students started to work in pairs. The subject polygonal numbers was new to the students and they had no prior experiences (from the classroom, according to the teacher) with this kind of tasks.



Figurtallene (polygontallene)

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Figurtall	Symbol	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	Generell n
Trekanttall	t_n	1	3	6	10	15	21	
Firkanttall	f_n	1	4	9				
Femkanttall	p_n	1	5					
Sekskanttall	h_n	1	6					
Syvkanttall	s_n	1	7					
Åttekanttall	o_n	1	8					
k-kanttall	k_n	1	k					

Figure 6

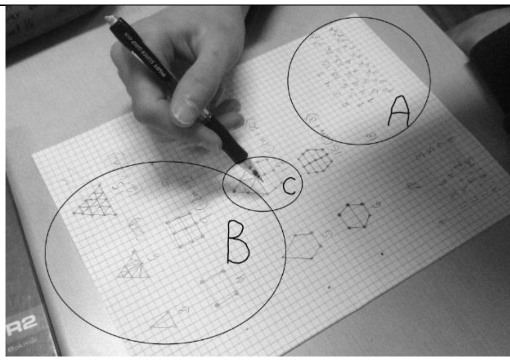
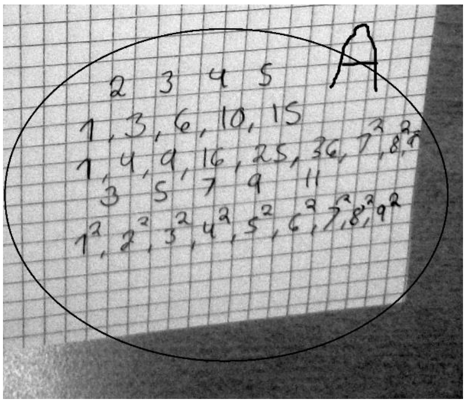
There was no restrictions on what methods they might to use, but the teacher gave no hints or sub questions, neither. One strategy for completing the form would be to study the pattern of increase, as shown in Figure 7.

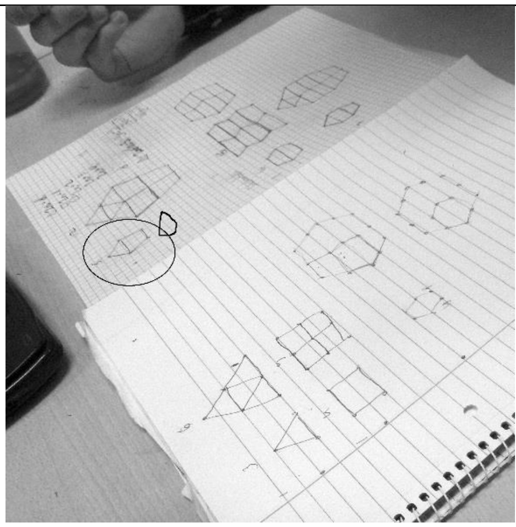
		n=1	n=2	n=3	n=4	n=5	n=6	General
Triangular Numbers		**1**	**3**	**6**	**10**	**15**	**21**	**½ · n(n+1)**
	Increased by	2	3	4	5	6	n	
Square numbers		**1**	**4**	**9**	**16**	**25**	**36**	**n^2**
	Increased by	2 · 2 - 1	2 · 3 - 1	2 · 4 - 1	2 · 5 - 1	2 · 6 - 1	2 · n - 1	
Pentagonal numbers		**1**	**5**	**12**	**22**	**35**	**51**	**$3/2\,n^2 - ½n$**
	Increased by	3 · 2 - 2	3 · 3 - 2	3 · 4 - 2	3 · 5 - 2	3 · 6 - 2	3 · n - 2	
Hexagonal numbers		**1**	**6**	**15**	**28**	**45**	**66**	**$2n^2 - n$**
	Increased by	4 · 2 - 3	4 · 3 - 3	4 · 4 - 3	4 · 5 - 3	4 · 6 - 3	4 · n - 3	
k-polygonal numbers		**1**	**k**	**3(k - 2) - (k - 3) + k**	**…**	**…**	**…**	**$½\,k\,(n^2 - n) - n^2 + 2n$**
	Increased by		3(k - 2) - (k - 3)	4(k - 2) - (k - 3)	5(k - 2) - (k - 3)	6(k - 2) - (k - 3)	n(k - 2) - (k - 3)	

Figure 7

Most of the students combined drawings with counting and, simultaneously, looked for patterns in the rows and/or columns containing the numbers obtained from the drawings.

Figure 8 entails the transcription of the episode (my translation) in the left column and my reflections and explanations in the right column:

	Episode 2 (Danielsen 18.03.2014 video8)	
1	The two students B1 and B2 sit and work together. They have already managed to write the first five triangular numbers 1, 3, 6, 10 and 15 and the square numbers 1, 4, 9, 16, 25, 36, 7^2 8^2 and 9^2 (Area A Pictures a and b) based on their drawings (Area B Picture a). Apparently, their unarticulated plan was to find a pattern for the extension from triangular numbers to square numbers, which they could extend to create the pentagular numbers and, afterwards, the succeeding polygonal numbers. B1:..*Then the next one is seven squared,* (writes 7^2),*the next one is eight squared* (writes 8^2), *the next is nine squared* (writes 9^2 in Area A Pictures a and b),	 ***Picture a*** 

		Picture b
2	B1: *then we know the difference between these* (points to the square numbers, points to the numbers 3, 5, 7, 9, 11 in area A Pictures a and b)	*Their preliminary choice of a strategy was, apparently, to read a pattern from the increase of the square numbers. The teacher's introduction had lead them in this direction (without giving any details, though)*
3	B1: *.. so in fact you have* (writes 1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^2, 9^2, last line in area A, Pictures a and b)	*B1 rewrites the square numbers in powers of two, apparently for making it easier to read a pattern*
4	B2: *But we cannot..* B1: *How can we write a formula for this?* B2: *For the triangle, it is not squared at least* B1: *But the triangle is different (..)*	It becomes clear to B1 and B2 that the pattern they look for cannot be as simple as an increase in powers
5	B1: *The triangle, it is something with its three sides, with the triangle in the middle somehow..* (points to area C, Picture a) B1: (draws a triangle, covered by his hand on Picture a)	30 seconds silence B1 and B2 are both starring at their drawings. According to my interpretation, they reconsider the strategy and try to take inspiration for a new strategy
6	B1*:(..) one more. What is the formula for the square?* B2: *Yes I see that, the square is okay. But..*	This sounds as if B1 still considers the old strategy of extension, and maybe he wants to check it out again. B2 is finished with the squares and he does not reconsider the same extension idea
7	B2: *But then, the triangle, you can somehow..* (points to the polygon in area D on his drawing Picture c)	 ***Picture c***
8	B2: *For example, for the pentagonal, then you may in a way, you can take a formula for the triangle and a formula*	B2 takes inspiration from his drawing to express the pentagonal numbers by a formula, which he can create by adding the

	for the square and add them	formulas they already know.
9	B1: *Then you can use it for all* B2: *Yes you can do it with all of them* B1: *Yes, exactly. So This is the square* (points to the squares in area B in Picture a), *that is why it becomes like this ..*	B1 acknowledges that the principle is applicable for all the pentagonal numbers and B2 agrees. B1 recognises the squares as parts of the pentagonal numbers in the first few cases in his own drawings
10	B1: *For example these points here, they have two in common..* B2: *Yes yes..*	They start to figure the formula out as a sum when taking into account that the triangle and the square has one line in common, according to the drawing

Figure 8

Discussion and Conclusion

The discussion in (Andresen 2015a) of episode 1 concludes that the episode contains an example of the algorithmic reasoning (AR) described by Lithner (2008), first Column Figure 3, where the students copy the solution of the preceeding task. It also contains examples of initial creative thinking in the second and third column, Figure 3. All the episode's examples of creative thinking concern elementary reasoning rather than problem solving but they took place in a problem solving setting.

In parallel, the discussion in (Andresen 2015b) of episode 2 compares the episode with Lithner's criteria for CMR (Lithner 2008): The four criteria for CMR were fulfilled in the case, only if shifts between different representations can be seen as part of the 'mathematical foundation' in the third one. The GMC in episode 2 was founded on the students' competence in shifting between the different representations (numbers, formulas and drawings) of the polygonal numbers. Their arguments for supporting the strategy choice were anchored in both students' representational literacy, which is an aspect of meta representational competence (MRC) as it was described above.

One of the students' representational literacy was revealed in the episode's scheme row 8, where B2 talked about the formula for the triangle and for the square, and about adding these two, without even to discern between the different representations. The other student immediately understood the idea. Episode 2 illustrates how the experimental lesson on polygonal numbers, founded on interplay between different representations, could be supportive of the students' development of meta representational competence as well as their creative, mathematically founded reasoning.

The student's creative reasoning in both episodes happened in a glimpse. In episode 1, it happened when the students E1 and E2 realised that they had squares and square numbers. In episode 2, it happened when the student B2 caught the connection between the pentagon consisting of a triangle (with a corresponding formula) and a square (with a corresponding formula) on the one hand, and, on the other hand, the algebraic number of which he wanted to have a formula.

Hence, in both episodes the GMC happened in the moment, when the student managed to see the two as different representations of the same object. A number of episodes from our data contains examples of GMC, which happen in a similar way when a student manage to establish a link between two different representations of the same object. Further analysis of data from our group's experiments may provide interesting insight into the connections and relations between CMR, GMC and MRC.

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PISA 2012 – Performance in Mathematics and School Size

Kristín Bjarnadóttir[a] – Freyja Hreinsdóttir[a*]

[a] *School of Education, University of Iceland, Sæmundargötu 2, 101 Reykjavík, Iceland*

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Abstract

A study of the results of PISA 2003 in Iceland showed that pupils in the two largest schools did was significantly better than in smaller schools. The score was particularly low in schools with 11–25 participants in PISA. A study of the PISA 2003 score in Denmark also showed better results in larger schools than in smaller ones. When the results of PISA 2012 in Iceland were published, the Educational Testing Institute of Iceland was asked to classify the results into four categories based on school size. The results in the largest schools turned out to be significantly better than in smaller schools. A questionnaire was sent to a selection of schools in each category, omitting the category of the smallest schools. In the questionnaire mathematics teachers were asked questions on their education, proportion of their work in teaching mathematics, experience in teaching mathematics in lower secondary school and material used. The results indicated that full-time work in mathematics teaching, many years of teaching mathematics and in particular continuity in teaching i.e. teachers' experience in teaching the same group and the same material for many years leads to better performance.

Keywords: Mathematics. Teacher education. Teaching experience. PISA 2012. School size.
Classification: B50, D10

Introduction

What is PISA?

PISA is an international survey into the competence and skills of 15 years olds in reading, science, mathematics and problem solving. PISA is an abbreviation of Programme for International Student Assessment. The survey is done by OECD and totally 65 nations participated in PISA 2012. The Educational Testing Institute of Iceland was responsible for the PISA survey on behalf of the Ministry of Education (Halldórsson, Ólafsson and Björnsson, 2012).

Mathematical literacy

The theoretical framework of PISA is based on the concept literacy which means the skills of students in

- drawing conclusions from what they know
- use their knowledge in new circumstances
- analyze, discuss and express their ideas when interpreting information and solving problems in different circumstances

*Corresponding author: freyjah@hi.is
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The concept literacy refers to the ability of students to use their knowledge and competence in key subjects and to analyze, understand and express in an efficient way solutions to various problems in many different circumstances. To acquire literacy is a lifelong process, which not only occurs in school but also through interaction with family, peers, colleagues and by participating in social activity. Mathematical literacy in PISA measures the ability of individuals to state, use and interpret mathematics in many different ways. It is the ability to reason mathematically and use mathematical concepts, methods, facts and tools to describe, explain and predict various phenomena (Halldórsson, et al., 2012).

The content of the PISA problems evolves around four main ideas on numbers, algebra and geometry. They intersect and relate in many ways to

- quantity
- shape and space
- relation and change
- uncertainty and data

It is expected that students can master calculators when solving the PISA problems. When dealing with algebraic problems the emphasis is on creating formulas and generalizing, i.e. to use mathematical symbolic language but very little emphasis is on rewriting and simplifying. A large emphasis is placed on reading graphs, interpreting and relating to given information, interpret formulas and estimate the effect of changing the value of a variable, read from maps, follow directions and calculate distance and area using appropriate scales.

Proficiency levels

The proficiency of students in answering questions in PISA is divided into 7 levels, 0–6. Students at level 6 in PISA are able to solve the most difficult problems and get more than 669 points. Students at this level have mathematical thinking and deductive abilities at a high level. They are able to draw conclusions and use information based on their research and models to solve complicated problems and use their knowledge in new contexts. They can connect information presented in different ways and adapt it to various circumstances. These students use intuition and understanding together with exceptional skills in symbolic and formal mathematical operations and relations to develop new approaches and methods to deal with new circumstances. Students at this level are able to communicate their answers precisely together with their thoughts on their discoveries, interpretations and reasoning and are able to explain why certain operations are used to bring mathematical problems from daily life to a mathematical form.

Students below level 1 get 358 points or less. They can possibly solve simple mathematical problems like reading a clearly marked value from an illustration or a table where the marks correspond to words used in the introductory text and the question posed. In this way the choice is clear and the connection between the graph and description seems obvious. These students are also able to solve mathematical exercises with integers by following direct instructions.

The average result for students in OECD countries was 494 points but the average for Iceland was 493 points. Further information on results and level distribution can be found in (Halldórsson et al., 2012, pp. 18–19, 26–28).

Size categories of Schools in PISA 2003 and PISA 2012

The results of PISA 2003 in measuring mathematical literacy

The results from PISA 2003 indicated that the results were better in larger schools than smaller (Bjarnadóttir, 2008), see Figure 1.

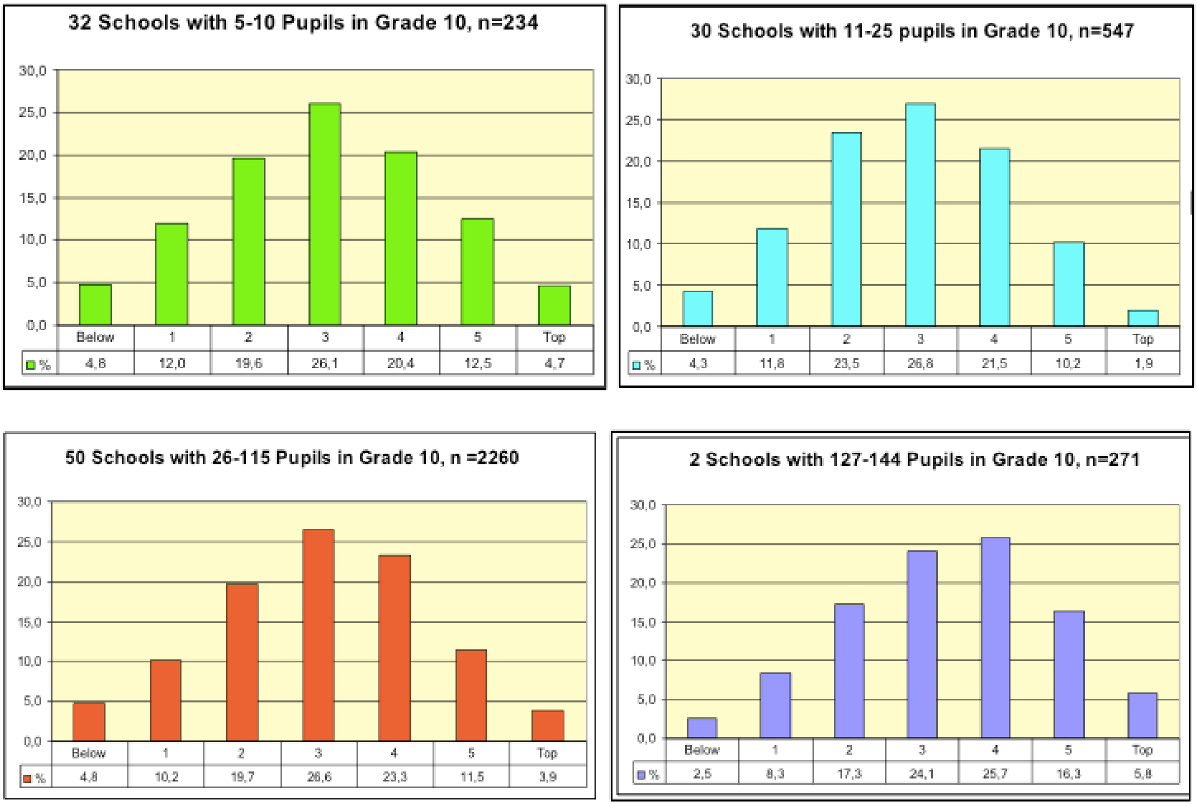

***Figure 1** Comparison of performance in mathematical literacy in PISA 2003 based on the number of participants divided into 4 categories.*

In Figure 1 it is shown that proportionally more students reached level four or higher in schools where 26–115 students participated in PISA than in schools with 11–25 participants where it can be assumed that each age group had only one class. The figure also shows that the result was proportionally better in the largest schools (only 2 schools were in that group).

Performance in PISA 2012 measuring mathematical literacy

The Educational Testing Institute of Iceland analyzed the performance in PISA 2012 in size groups in collaboration with the authors. The size groups were chosen as 1–10, 11–25, 25–40 and 41–128 participants in PISA 2012. This time it was decided to enlarge the group with the largest group from what was done for PISA 2003 (the largest group then only had 2 schools). This was done to see if the size of the school mattered and to minimize the effect of certain schools.

Not all students in every school participated in PISA. Exemptions were approximately 5% since it is generally expected that exemptions are only made for health reasons (Halldórsson et al., 2012). The numbers however indicate that the participation was less than 95%. In 2012 there were 4500 15 year olds in Iceland (Statistics Iceland, web) but 3509 students participated in PISA 2012 or 78%.

The results can be seen in Figure 2. The figure shows that the same pattern as in PISA 2003 is repeated in PISA 2012. Proportionally more students reach level 4 or higher in the group of schools with the highest number of participants. There are also proportionally more students that only reach level 1 in the schools with the fewest participants.

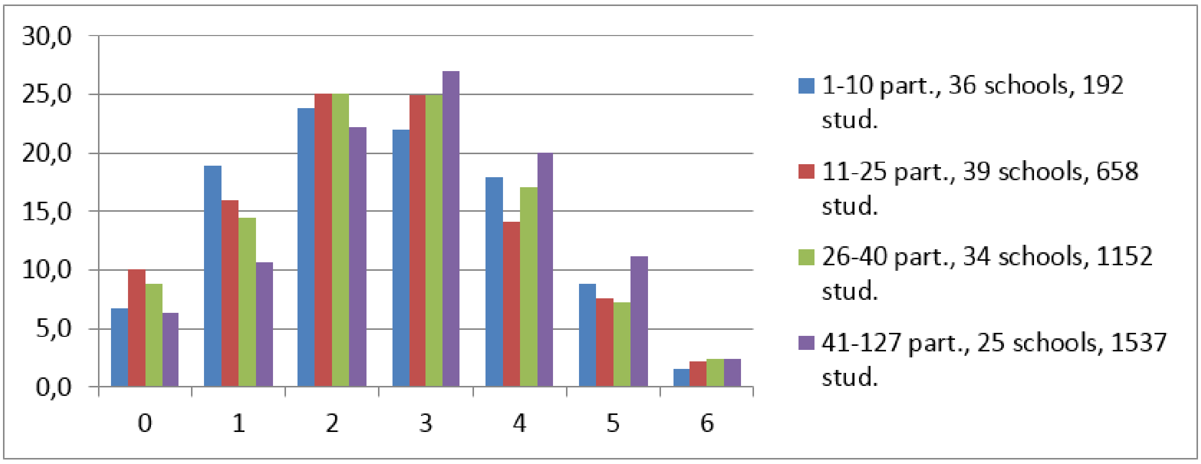

Figure 2 *Comparison of performance in Mathematical Literacy in PISA 2012 in different school size categories*

The difference is also clear when looking at the total score of the four groups:

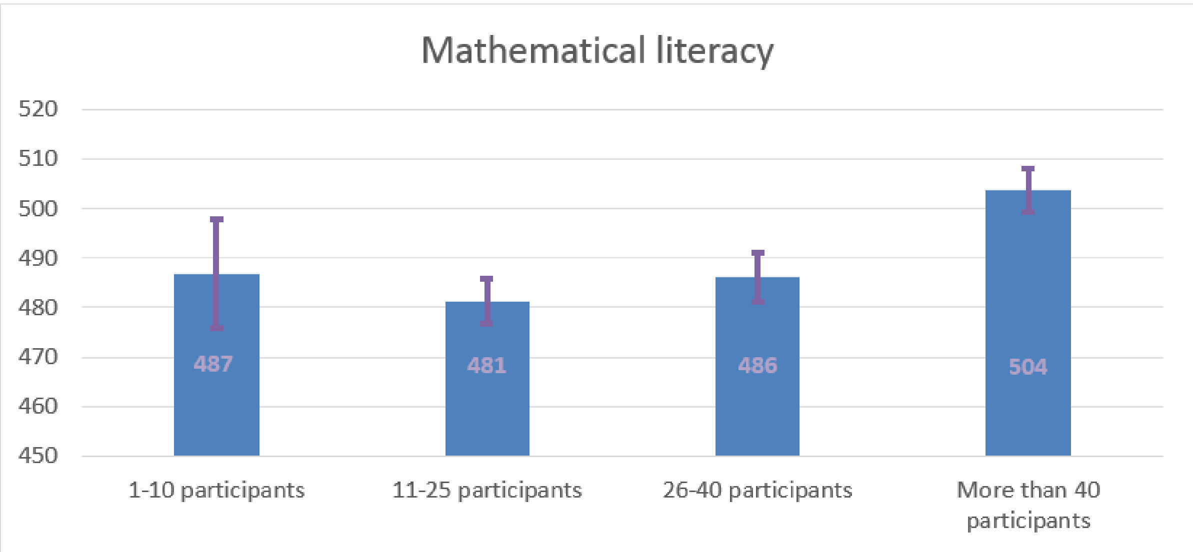

Figure 3 *Total score in Mathematical Literacy in PISA 2012 classified by the size of school (together with 90% confidence limits).*

Figure 3 shows clearly that the total score is significantly better in larger schools than in smaller ones. The average total score in the largest schools in 504 points, which is significantly better than the average score of Icelandic students, 493 points. In our research we try to examine the reasons for the better performance in the larger schools.

Theoretical background

Difference in performance in PISA 2003 in Denmark with respect to school size

Niels Egelund (2006) reached the conclusion on PISA 2003 in Denmark that the performance was better in larger schools than smaller. Egelund claims that the performance gets better with the size of schools up to 650 students (this is the total number of students in 10 age groups). (see Figure 4). He refers to international research, mainly American, but says that this research is not applicable for comparison with Danish schools. Firstly, these are not 10 year schools, secondly there are schools with very different situations from Denmark in areas of poverty and thirdly there are very few teachers in the United States that teach all age groups and many subjects while this is common in Denmark. The Icelandic school system is very similar to the school system in Denmark, and research is therefore comparable even if Denmark has very few small schools in sparsely populated areas. Egelund's theory is that large schools

- have broader competence of teachers across subjects and school levels
- have better possibilities for teachers to work as teams in each subject
- have better possibilities for teachers to teach the subjects they have specialized in
- have less effect of conflict between teachers

Figure 4 shows the relation between total number of students in Danish schools and average total score in PISA 2003.

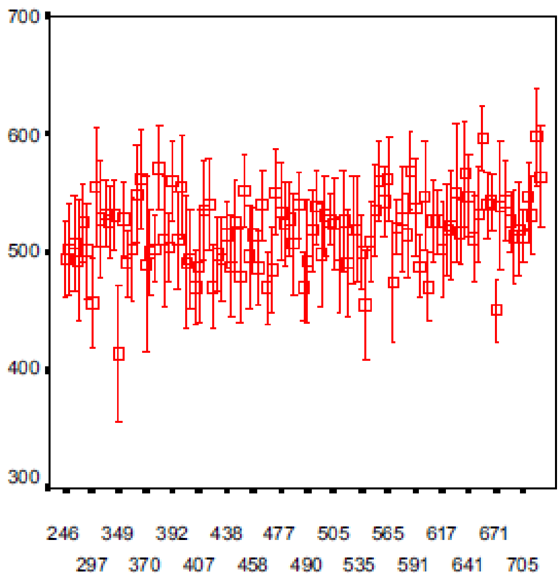

Figure 4 *Average total score in PISA 2003 (with 95% confidence interval) is the vertical axis and size of school is the horizontal axis. (Egelund, 2006, p. 311)*

Theories on subject matter knowledge and pedagogical content knowledge

Shulman (1986) defined several types of knowledge that teachers need to have. He distinguished in particular *content knowledge*, that is a deep understanding of the subject itself, and *pedagogical content knowledge*, which is to know the subject matter for teaching and to know the most useful forms of presenting it, explaining and demonstrating it. Thirdly, Shulman defined *curricular knowledge* as a necessary factor in teacher education; that is pedagogical knowledge behind the subject's curriculum and the material taught in other subjects as well as solid knowledge of the previous and future curricular content.

Shulman article is quite old and covers general subjects but not specifically mathematics. Some scholars have continued research in a similar direction. These include Krauss, Baumert, Brunner and Blum (2008) and Neubrand (2008). They claim there is a strong correlation between *content knowledge* (CK) and *pedagogical content knowledge*, PCK. Their view is that PCK is strengthened by strong CK but that CK is only one possible way to PCK and that emphasis on pedagogics in teacher education is another possible way.

More scholars have discussed the connection between subject knowledge and pedagogical knowledge, e.g. Barbara Jaworski. To capture the complexity of the many factors that need to be considered in teaching, she developed the concept *the teaching triad* (Jaworski, 1994). Jaworski based the idea of the teaching triad originally on research on the teaching of mathematics. That research showed that the teachers took into consideration three interconnected factors when organizing their teaching: *management of learning*, ML, *sensitivity to students*, SS and *mathematical challenge*, MC.

Jaworski considers the three components as tightly woven factors in the total commitment of the teachers who need to always weigh them against each other in their teaching. She describes them by the figure below.

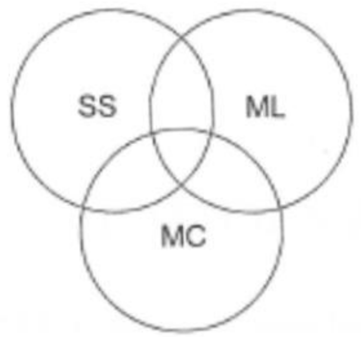

Figure 5 *The Teaching Triad (Jaworski, 1994, pp. 107–8)*

The theories of Shulman, Kreuss et. al., Neubrand and Jaworski all point in the same direction: a teacher needs to have subject knowledge in order to provoke the thought of students, he needs *pedagogical content knowledge* on the subject to be able to ask the right questions, he needs to know and take into consideration *the curriculum* and know what has already been studied and what is to be studied, and he needs to be *sensitive to his students needs*.

The investigation

Choice of schools

We contacted teachers from ten schools in each size group. Teachers in two schools didn't want to participate, but in one of those two cases (with 11 – 25 participants) we were able to find another school where teachers were willing to participate. One teacher in the size group 25–40 did not answer questions on education but all other questions.

We decided not to include the smallest school in the research. These are schools where the number of participants in PISA 2012 was between 1 and 10. This was done since we expect that the situation in these schools is different from other School Groups, several age groups may be taught together, the total number of pupils (192) is irrelevant in comparison to the total number of participants, and the teaching is adapted to the individuals in those schools.

The number of students in each group is as follows:

Table 1 *Number of students in the chosen schools*

Groups of schools	**11–25 students**	**26–40 students**	**41–130 students**	**Total**
Total number of schools	39	34	25	98
Number of chosen schools	10	9	10	29
Number of chosen schools in Reykjavík and surrounding area	5	6	6	17
Total number of students	658	1122	1537	3317
Total no. of stud. in chosen schools	174	303	721	1198
Percentage	26%	27%	47%	36%

In the year 2012 the population of Iceland was 319.575. Of those, there were 4500 15-year olds (Statistics Iceland, web) but totally 3.509 students participated in PISA 2012 or 78% as was stated earlier. In the capital region, i.e. Mosfellsbær, Reykjavík, Seltjarnarnes, Kópavogur, Garðabær, Álftanes and Hafnarfjörður 193.444 people lived or 61% of the total population, while the number of 15-year olds was 2.647 or 59% of the total number of 15-year olds. Choosing 5–6 schools in each size group from the capital region reflects therefore well on that area versus the whole country.

The goal of the investigation, research questions and methods

The aim of the research is to investigate whether the education, experience and specialization of teachers, or teaching material used, affected the results of the students. The following research questions were posed in the different School Groups:

1. Is it possible to detect a difference in the educational specialization of the teachers?
2. Is it possible to detect a difference in the experience of the teachers?
3. Is it possible to detect a difference in the specialization of the teachers in their work?
4. Is it possible to detect a difference in the use of textbooks and other teaching material?

A letter was sent to the principals of the 30 schools. It was not considered necessary to get their permission for the investigation since the results from individual schools were not published and the teachers themselves decided if they wanted to take part or not. After gathering information on who had taught the student groups considered, a questionnaire was send via email followed by a telephone interview in the next few days. The questionnaire consisted of the following items:

- the education of the teachers
- which year they had been teachers of the PISA 2012 student group
- which proportion of their work was the teaching of mathematics during the school year 2011–2012

- for how long they had been teaching mathematics in the same proportion
- what teaching material was used for the PISA 2012 group
- whether any of the students had upper secondary level mathematics as an elective course (STÆ103).

The investigators came across some hindrances in their work in locating teachers:

- schools had merged so some schools no longer existed
- teachers had moved, some of them abroad
- teachers did not remember if they had taught the group in question since two years had passed and even principals did not know which teachers to contact.

The information that was easiest to obtain was from interviewing teachers who had been working in the same place for a long time or were heads of departments. Sometimes, there was some delay in getting a hold of the teachers. The interviews took place in March, April and May in the spring of 2014.

Results

Basic information

The number of students and teacher, the average number of students per teacher and the average age of teachers in the schools that were contacted was as follows:

Table 2 *Number of students and teachers, students per teacher and average age of teachers*

	Number of students	**Number of teachers**	**Students per teacher**	**Average age of teachers**
School Group 4 > 40 participants	721 (47%)	25	29	48 years
School Group 3 25–40 part.	334 (30%)	21	16	50 years
School Group 2 10–25 part.	174 (26%)	17	10	42 years
Total	1229 (35%)	63		47 years

From the table we see that teachers of approximately half of students in schools with more than 40 participants were contacted. On the average each one of them had contact with 29 students so many have been teaching more than one class.

Education of the teachers

The majority of the teachers in all groups had completed a B.Ed.-degree with mathematics education as a special subject or had taken extra courses in mathematics education after finishing their first degree. Three teachers, all in School Group 2, had an old degree (from the time when teacher education was at the upper secondary level), which they had later complemented with additional education. They are classified as “other education”. Four teachers had finished their B.Ed. from the University of Akureyri, with specialization in science teaching. They are classified with the mathematics specialization group.

Table 3 *The education of the teachers*

	Number of teachers	B.Ed or M.Ed. in Mathematics Education		B.Ed. with additional courses in Mathematics Education	Other, with additional courses in Mathematics Education	B.Ed. Other spec.	Other studies
		M.Ed.	B.Ed.				
School Gr. 4 >40 part.	25	3	9	4	1	6	2
		48%		16%	4%	24%	8%
School Gr. 3 25–40 part.	20	2	12	1	0	5	0
		70%		5%	0%	25%	0%
School Gr. 2 10–25 part.	17	0	10	0	1	3	3
		59%		0%	6%	17%	17%

It seems that the education of teachers is the highest in School Group 3 when considering mathematics education and mathematics. It is hard to distinguish between the education of teachers in School Group 4 and School Group 2. No teachers had completed a B.Sc. degree in mathematics.

School years that the teachers taught the PISA 2012 group

Teachers in School Group 4 had the highest occurrence of teaching the students during all three years of lower secondary school but the teachers in School Group 2 had the lowest occurrence.

Table 4 *School years that the teachers taught the group*

	Number of teachers	2011–2012	2010–2011	2009–2010
School Gr. 4 >40 participants	25	23 (92%)	24 (96%)	19 (76%)
School Gr. 3 25–40 part.	21	19 (90%)	17 (81%)	13 (62%)
School Gr. 2 10–25 part.	17	12 (71%)	11 (65%)	10 (59%)

The proportion of mathematics teaching in the teachers' work

The majority of the teachers in School Group 4 worked full time as mathematics teachers, possibly with some administrative duties as part of their work.

Table 5 *Proportion of mathematics teaching in the teachers' work*

	Number of teachers	Full time	>50%	<50%
School Gr. 4 >40 participants	25	16 (64%)	7 (28%)	2 (8%)
School Gr. 3 25–40 part.	21	9 (43%)	11 (52%)	1 (5%)
School Gr. 2 10–25 part.	17	8 (47%)	8 (47%)	11 (6%)

The number of years the teachers had taught mathematics

Table 6 *The number of years the teachers had taught mathematics*

	Number of teachers	1 year	2 years	3 years
School Gr. 4 >40 participants	25	0	1 (4%)	24 (96%)
School Gr. 3 25–40 part.	21	0	2 (10%)	19 (90%)
School Gr. 2 10–25 part.	17	4 (25%)	2 (13%)	10 (63%)

Teachers in School Group 2 had the least experience in teaching mathematics. They also had the lowest average age, 42 years, while the average age was 48 and 50 in School Groups 4 and 3, respectively. It was clear that change in mathematics teacher for the student was most common in School Group 2. If teachers took a leave of absence or quit (due to maternity or studies or other reasons) he or she was usually replaced by an outsider where as in the larger schools it seemed to be easier for one teacher in the group to take on the teaching of an absent one.

Upper secondary level mathematics

It is common that students are offered the possibility to study the first course in upper secondary level mathematics STÆ103, as an elective course. This turned out to be more common in School Group 4 with the highest number of PISA 2012 participants, see Table 7. Generally about 20 – 30% of the students chose the course.

Table 7 *Upper secondary mathematics course offered*

	Number of school that offered STÆ103 as an elective course
School Gr. 4 > 40 part.	8 of 10 schools or 80%
School Gr. 3 25–40 part.	7 of 9 schools or 77%
School Gr. 2 10–25 part.	7 of 10 schools or 70%

Textbooks

There are two possible series of textbooks to use: one of them is called *8 – Tíu (8 - Ten)* and the other one *Almenn stærðfræði (General mathematics)*, see Table 8. Most of the schools in all groups use both series; one as a main text and the other one as supplementary material.

Table 8 *Textbooks used*

	8 –Tíu		*Almenn stærðfræði*		Both series
	Main	Suppl.	Main	Suppl.	
School Gr. 4 >40 part.	**5**	2	**2**	4	**3**
School Gr. 3 25–40 part.	**6**	1	**2**	5	**1**
School Gr. 2 10–25 part.	**3**	2	**3**	2	**4**

Summary

It is of great concern that students who perform at level 6 in Iceland are fewer in PISA 2012 than in PISA 2003. When comparing the education of the mathematics teachers in the three School Groups it seems that teachers in School Group 3 have the most extensive education in mathematics or mathematics teaching. It was most common in School Group 4 that the teachers had taught the group for all three years of lower secondary school but this was least common in School Group 2. Many of the teachers had all or more than half of their teaching duties in the teaching of mathematics but this was most common in School Group 4. Teachers in School Group 4 had most experience but teachers in School Group 2 had least experience.

It was most common to offer upper secondary mathematics as an elective course in School Group 4; this was however done in the majority of all schools. The textbooks used were similar. The majority used *8-Tíu* or both textbook series. This indicates that teachers let curriculum direct their teaching more than the textbooks.

Conclusion

Figures 2 and 3 show that the performance is significantly better in large schools than in small schools. We sought explanations for this by investigating the teachers' education, experience, continuity in the teaching of the PISA 2012 group, textbooks used and students possibilities in choosing upper secondary mathematics. Our investigation did not show that difference in the teachers' education, see Table 3, could explain this and thus answer research question 1.

Investigation on the teachers' experience is two-sided; both how much they had taught the particular group who took the PISA 2012 test and how many years they had been teaching. The years of teaching the PISA 2012 group are given in Table 4. Almost every teacher in School Group 4 had been teaching the group both in their 9th and 10th year, most of them in the 8th year as well. It is also clear from the table that many of them had been teaching more than one groups which means they had to repeat their teaching. They therefore had more opportunities to contemplate the material and their teaching and thus improve it. The proportion was the lowest in School Group 2. This indicates that teachers in large schools have the most possibilities to acquire broad competence across material of consecutive years, which Egelund (2006) claims as an explanation of the better results in the largest schools. This is also in harmony with Shulman's (1986) theory on the expected curricular knowledge of these teachers. Table 6 shows that the teachers in the largest schools had also had the most experience in teaching of mathematics see research question 2, but that supports the same theories.

It can be read from Table 5 that the largest proportion of teachers in the School Group 4 were full-time math teacher, but research question 3 was on the specialization of teachers in their work. It supports the hypothesis of Egelund that large schools provide the best opportunities for teachers to teach the subject they are best prepared for. It is also Egelund's assumption that large schools have the greatest potential for teachers to build professional teams.

It is not directly possible to read from the tables a difference in the depth of pedagogical content knowledge of teachers according to the theories of Shulman and Krauss et al. (2008) It can be assumed, however, that teachers who have taught for a long time have collected useful ways to present the material and are better at presenting the material in a way that is comprehensible to others. According to Table 6, teachers in School Group 4 have the longest experience but teachers in the School Group 2 shorter experience on average than teachers in the Schools of groups 3 and 4. It can also be assumed that teachers, who have lasted long in the job, are better at taking into account the learning conditions and needs of their students, professional and non-professional, and to challenge students professionally as Jaworski (1994) believes will lead to successful teaching, although nothing will be said with certainty from the above survey.

From the above it can be concluded from the information obtained from teachers in 29 of the 98 compulsory schools in Iceland with more than 10 participants in PISA 2012, that the experience of teachers, and especially the experience which comes from teaching the respective student group long and often, weigh the most to cause that the best results are achieved with students in the largest schools. Teachers in these schools seemed to have the best opportunities to get to know students and their needs and to have a good overview of the curriculum, both what came before and what will follow.

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Some Famous Problems of Discrete Geometrie

Vojtech Bálint*

Department of Quantitative Methods and Economic Informatics, Faculty of Operation and Economics of Transport and Communications, University of Zilina, Univerzitná 1, SK - 010 26 Žilina

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Abstract

Newton number, Kepler's conjecture, Happy End Problem and some packing and structural problems will be discussed. Highlighted are open problems.

Keywords: Packing, covering, points, lines, circles.

Classification: 05B40, 52C35

1 Introduction

For the problems of discrete geometry is symptomatic their clearness. The formulation of the problem is very simply and the understanding obviously requires no special knowledge. However, the solution often requires certain dose of ingeniousness and the method of the solution changes from the problem to the another problem. Obviously, my choice of the problems is subjective and many articles and books have been written about most of them. Many of them can be found in the references and the substantial part of those is easily attainable. For this reason I will present here only brief information about selected problems.

2 Newton number

Newton number (also kissing number) $N(B^d)$ in $d-$dimensional space is defined as the maximum number of non-overlapping unit spheres that can be arranged such that they each touch another given unit sphere. Trivially, $N(B^1)=2$ and easily can be proved that $N(B^2)=6$. In the dimension 3 a famous discussion between Isaac Newton and David Gregory took place on the campus of Cambridge University in 1694. The right result $N(B^3)=12$ was correctly proved by Schütte and van der Waerden [ScvW53] only in 1953 and a little later very elegant proof was published by Leech [Le56]. Extremely symmetric lattices found Leech [Le64], Musin [Mu03] proved $N(B^4)=24$. On the base on this and [OdS79], [Lev79] we have $N(B^8)=240$ and $N(B^{24})=196\,560$. No other precise value of $N(B^d)$ is known.

For other contexts see also [An04], [Bö03], [Bö04], [EdRS98], [De72], [EriZ01], [Had55], [Had57], [Zo98], [Hor75], [BMP05].

*Corresponding author: vojtech.balint@fpedas.uniza.sk
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3 Kepler's conjecture

Density of any arrangement of non-overlapping equal circles in the plane is at most $\pi / \sqrt{12}$ and in the densest arrangement the centers of the circles form the hexagonal lattice (incomplete proof in [Th892], corrected in [Th10]). Similar question in E^3 for the densest packing of equal balls is much more complicated.

Kepler's conjecture [Ke611]: Arrange the balls such that their centers form the hexagonal lattice in the plane. The densest packing of balls one get by translations of such planar arrangement.

Gauss [Ga831]: among the *lattice* packings the above mentioned arrangement is the densest.

So, to prove the Kepler's conjecture it is necessary to show that any *irregular* arrangement has the density less than the above mentioned regular arrangement. In this reason Kepler's conjecture became the longest unsolved problem of combinatorial geometry.

Fejes Tóth László [Fe53] showed that to obtain the maximum packing density this can be reduced to a finite though very large number of calculations.

Rogers [Ro58] showed almost sharp value of the density.

Hales [Ha92] – it is sufficient to find the extremum of the function of 150 variables.

Hsiang [Hs93]: I have a direct geometrical proof of the Kepler's conjecture.

Gábor Fejes Tóth (son of L. Fejes Tóth) in his referee report on Hsiang's paper: „As far as details are concerned, my opinion is that many of the key statements have no acceptable proofs."

Controversy between Hsiang and Hales [Ha94], [Hs95].

Hales [Ha97], [Ha97b], [Ha00] reported that his proof is complete. But his proof (based on the idea of Fejes Tóth László) had 250 pages and included a 3-gigabyte computer program, because it was necessary to analyze more than 5000 configurations of spheres.

Non-standard review, 12-membered panel: András Bezdek, Michael Bleicher, Károly Böröczky, Károly Böröczky, Jr., Aladár Heppes, Wlodek Kuperberg, Endre Makai, Attila Pór, Günter Rote, István Talata, Béla Uhrin, Zoltán Ujváry-Menyhárd. In 2003, the head of the panel of reviewers, Gábor Fejes Tóth, announced that a group of reviewers is 99% certain correctness of the proof, but - of course - cannot confirm the accuracy of all computer calculations.

Detailed proof can be found in [HaF06] and [FeH11].

Results for $d-$dimensional space, $d \geq 4$, can be found e.g. in [Var95], [CoS96], [Ro58], [Be02a], [CoE03].

4 Happy End problem

Find the smallest integer $f(n)$ such that every set of $f(n)$ points in general position in the plane contains the vertices of some convex $n-$gon.

Conjecture. (Erdős – Szekeres [ErS35]) Every set of $2^{n-2}+1$ points in general position in the plane contains a convex $n-$gon, so $f(n)=2^{n-2}+1$.

Theorem. ([ErS35]) For each integer $n\geq 4$ there is the smallest number $f(n)$ such that from any $f(n)$ points in general position in the plane one can choose n points such that they form the vertices of a convex $n-$gon.

Only estimates are known: $f(n)\leq \binom{2n-4}{n-2}+1$ (see [ErS35]) and $2^{n-2}+1\leq f(n)$ (see [ErS60] – this lower bound is perhaps the best possible one). See [ChuG98], [KlP98] and [TóV98] for small improvements of the upper bound.

Conjecture. ([Er78]) From a sufficiently large number $G(n)$ points in general position in the plane is always possible to choose n points such that they are the vertices of an empty convex $n-$gon, so such $n-$gon which contains no other point of the given $G(n)-$point set in his interior.

Natural task is to find the smallest possible value of $G(n)$.

Trivially, $G(3)=3$, $G(4)=5$. Harborth [Har78] proved $G(5)=10$. Surprisingly, Horton [Ho83] for every n constructed such set of n points that does not contain the vertices of an empty convex $7-$gon, and thus proved the result $G(7)=\infty$.

Estimates for the value $G(6)$ are in [Va92], [Ov03], [Ni07], [Ge08], [Kos07] and generalizations in [BiF89], [BiF89b], [BK01], [PaT00], [PóV02].

5 Sylvester – Gallai theorem

Sylvester [Sy893]: „Is it true that any finite set of points in the Euclidean plane, not all on a line, has two elements whose connecting line does not pass through a third?“ Such connecting line is called ordinary line. The number of ordinary lines determined by finite noncollinear point set we denote by l_2.

Theorem. (Sylvester – Gallai) Non-collinear finite point set determines at least one ordinary line.

The most beautiful proof comes from L. M. Kelly; Kelly's proof is published in [Cox48] and also in excellent book [AiZ04] *Proofs from THE BOOK*.

Conjecture. ([Di51], [Mot51]) For every $n\neq 7,13$, the number l_2 of ordinary lines determined by n noncollinear points is at least $n/2$.

Theorem. ([KeM58]) $l_2\geq \frac{3}{7}n$.

Hansen ([Han81]): incomplete proof of the conjecture $l_2\geq n/2$.

Theorem. ([CsS93]). If $n\geq 8$, then $l_2\geq \frac{6}{13}n$.

For $n=2m$ Motzkin found configurations with exactly $\frac{1}{2}n$ ordinary lines. So, the lower bound $n/2$ in the conjecture cannot be enlarged. For odd n the best known configurations (Böröczky) determine at least $\frac{3}{4}n$ ordinary lines.

Theorem. ([GrT13]) If n_0 is sufficiently large, then $l_2\geq n/2$ for every $n\geq n_0$.

Despite the fabulous result of Green and Tao the problem is not entirely closed. In particular: how large is n_0 ?

The history of the solution of Sylvester's problem is a little confused and complicated, probably due to the 2nd World War. A good overview can be found in [BMP05], but for details we can recommend [Er83], [Er43], [St44], [Mel 41], [Bor83], [BorM90], [BrE48], [Cox89], [EHK63], [Ed70], [HeK60], [Ku72], [La55], [Li88], [Mot51].

Related problems have enormous literature. For further contexts and especially for many generalizations (besides other to pure combinatorial access, e.g. number of circles, horocycles, planes, unit circles, ...) see [AgA92], [Bá79], [Bá*94], [Bá98], [Bá99], [Bá06], [Bá07], [CrM68], [Brk72], [St44], [Hh55], [BrE48], [CFG94], [BáČ07], [KoP60], [Grü99], [BorM90], [Ch70], [ErP95], [PaP00], [Han65], [Han80], [Mot51], [Pu86], [BMP05], [Tu77], [SzT83], [PaT97], [Pa*04], [Sz97], [Grü72], PaS04], [Al02], [Bá90b], [Bá*95], [Bá*97], [BáK01], [Bá03b], [BáB07], [Be99], [BFT01], [Ele84], [El67], [GRS90], [Har85], [HarM86], [JaH83], [Ju70], [KlW91], [Ni05], [PPS04], [Pi02], [Pi03], [Ra30], [Sco70], [SSV05], [TaV06], [TóV05], [Bá13], ...

6 Finite containers

Auerbach, Banach Mazur and Ulam proved that for every positive integer d and for every $V > 0$ there exists a number $f_d(V)$ such that each system of $d-$dimensional convex sets $M_i, i \in J$ with diameter at most 1 and with a total volume at most V can be packed into $d-$dimensional cube with side of length $f_d(V)$.

Problem. What is the smallest possible value of the number $f_d(V)$?

To answer this question it will be extremely difficult. For *parallel* packing of *boxes* the first upper estimate of the number $f_d(V)$ found Kosiński [Ko57]. This estimate improved Moon and L. Moser [MoM67]. For the dimension $d = 2$ better estimates found Meir and L. Moser [MeM68].

Question. Is it possible to pack all the rectangles $R_i = \frac{1}{i} \times \frac{1}{i+1}$, $i = 1, 2, 3, \dots$ into the unit square?

Question. Is it possible to pack all the squares with sides $\frac{1}{2i+1}$, $i = 1, 2, 3, \dots$ into the rectangle with area $\frac{1}{8}\pi^2 - 1$?

Both of previous problems are well grasp because specific systems of boxes are packed. Good estimates are in [Je94] and [Je95], better estimates in [Bá90], [Bá90b], [Bá98b] and almost sharp estimates (using a computer) in [Pau98]. But the precise answers are still not known.

Problem. ([MoM67]) Determine the least number A such that every system of squares with total area 1 can be parallely packed into some rectangle with area A.

Kleitman and Krieger [KlK70] proved that every such *finite* system can be packed into the rectangle with sides 1 and $\sqrt{3}$ and this upper estimate they improved in [KlK75] to the rectangle with sides $\frac{2}{\sqrt{3}}$ and $\sqrt{2}$; so, they proved $A \le \frac{4}{\sqrt{6}} \doteq 1{,}632\,993\,162$. Novotný [No95] showed nontrivial lower bound $A \ge \frac{2+\sqrt{3}}{3} > 1{,}244$ for the system of three squares with length of side $\frac{1}{\sqrt{6}}$ and one square with length of side $\frac{1}{\sqrt{2}}$. Because of small squares can be packed very economically, it is almost sure that exactly this configuration gives the extremum.

Conjecture. $A = \frac{2+\sqrt{3}}{3}$.

Nontrivial upper estimate of Kleitman and Krieger for the finite systems of squares improved Novotný in [No96] to $A < 1{,}53$ and in the paper [No99] he showed that for every five-element system of squares the equality $A = \frac{2+\sqrt{3}}{3}$ holds. Just this equality he showed also for 6, 7 and $8-$element systems in [No02]. This significantly strengthens the belief in the validity of the conjecture that the above mentioned 4-element system of squares is extremal. The upper estimate only recently improved (using a computer by a suitable discretization) Hougardy [Hou10] to $A \leq 1{,}4$, but also this estimate is still far from the expected exact value $A = \frac{2+\sqrt{3}}{3}$.

There are many open questions in higher dimensions.

* * *

In Malfatti [Ma803] one can found the problem of packig of three circles into the triangle such that the sum of area of packed circles is maximal. Surprisingly, this problem was solved only 191 years later, when Zalgaller and Los [ZaL94] proved that the maximum gives so called *greedy algorithm*. Sharp values for the packing of n equal circles into the equilateral triangle are known only for triangle numbers $n = \frac{j(j-1)}{2}$ ([Ol61]) and by Erdős forecasted result for $n = 14$ (see [Pay97]). Besides this there are known only some estimates reached by computer (see e. g. [GrL95]).

Exchange the triangle and circle gives the following problem.

Problem. ([AnM06]) Maximize the sum of area of three non-overlapping triangles which are packed in given circle.

The *greedy algorithm* does not give maximum for this problem. Andreescu and Mushkarov conjectured that the maximum area of $n \geq 3$ triangles packed in circle shall be achieved by dividing on triangles the regular $(n+2)$-gon inscribed in the circle. Bezdek and Fodor [BeF10] and independently also [Bá10] proved the weaker version of this conjecture: if all vertices of packed triangles are on the circle. A. Bezdek and Fodor [BeF10] noted that this problem seems *hopelessly difficult*. On the lecture we show a result reached recently by a computer.

* * *

For given integer $k \geq 2$ we are looking for the maximum radius $r = r(k) > 0$ such that k balls with radius r can be packed into the greater ball with radius $R > r > 0$.

Because of the position of the ball is precisely determined by the position of its centre, we can consider admissible arrangement of the centres of balls.

For the positive integer k denote $h(k)$ the greatest number such that into the circle with radius 1 can be packed k points such that the least distance of pairs of the packed points is $h(k)$. Sharp values of $h(k)$ for $k \leq 10$ showed Pirl [Pir69]. Besides this there are known only $h(11)$ (see [Me94]) and $h(12)$, $h(13)$, $h(14)$ and $h(19)$ (see [Fo00], [Fo03b], [Fo03] and

[Fo99]). The best known lower estimates of $h(k)$ for $n \le 65$ were found in [Gr*98] using a computer.

* * *

Given integer k, find the maximum radius $r = r(k)$ such that k smaller balls with radius r can be packed into the unite cube. In the plane there are known results for $k \le 28$ (see e.g. [NuÖ99], [Sc65], [ScM65], [Pe*92], [Ma04]). In [LGS97] one can found so called *billiard* algorithm for finding dense arrangements of centres of circles. Many of them are maybe the best possible one, but the proof of optimality absents. For a good survey see [AmB00].

Problem. ([Mo66], [Gu75], [Mo91], [MoP94], [BMP05]) Denote by $f(d)$ the maximum number of points which can be placed into $d-$dimensional unit cube C^d such that all determined distances of points are at leas 1. Find sharp values of $f(d)$ at least for small values of d.

Such arrangement of points we call admissible set. Trivially, $f(d) = 2^d$ for $d = 1,2,3$. Unicity of the packing in dimension 3 was proved in [BáB01] and [Sc66]. Besides this there is known only one precise value: $f(4)$, which was shown together with unicity of the configuration in [BáB03]. No other sharp values of $f(d)$ are known (see also [Bö04]).

Theorem. ([BáB03]) If an admissible set of $f(5)$ points contains all vertices of the unit cube C^5, then $f(5) = 34$. If an admissible set of $f(6)$ points contains all vertices of the unit cube C^6, then $f(6) = 76$.

Lower bounds (see [BáB03], [BáB07], [BáB08b]): $f(5) \ge 34$, $f(6) \ge 76$, $f(7) \ge 184$, $f(8) \ge 481$, $f(9) \ge 994$, $f(10) \ge 2452$, $f(11) \ge 5464$, $f(12) \ge 14705$. In [Horv10] the author constructed by Hamming codes admissible packings of $\frac{3^d + 2(d-1)3^{d/2} + 1}{2d}$ points into the $d = 2^k -$dimensional unite cube.

To obtain an upper bound is much more difficult. In [BáB08] we showed the upper bounds of $f(d)$ for $d = 6, \ldots, 12$. All this upper bounds was a little improved in [Ta10].

In [BáB12b] was proved $f(6) \le 120$, and this time this is the best known result.

In [BáB07] was proved the estimate $f(5) \le 44$; by another (very complicated) method Joós in [Jó08] showed the estimate $f(5) \le 43$, and he improved this in [Jó10] to $f(5) \le 42$. In this time the best known upper estimate is $f(5) \le 40$ (see [BáB12]).

Conjecture. $f(5) = 34$, $f(6) = 76$, $f(7) = 184$, $f(8) = 481$.

Asymptotic upper estimate $f(d) \le d^{d/2}\left(1 + \frac{1}{d}\right)^d \sim d^{d/2} e^{(1+o(1))\sqrt{d}}$ for d sufficiently large ([BáB03]) was improved on the base of writing communication [Ma] and using [FeK93] and [KaL78] to $f(d) \le d^{d/2} \cdot 0{,}63091^d e^{o(d)}$ in [BáB08]. Using [Ma] was shown also nontrivial lower estimate $f(d) \ge d^{d/2} \cdot 0{,}2419707^d \, \Omega\left(\sqrt{d}\right)$ (see [BáB08]).

7 Mix

If we take n points in the plane, $n \geq 4$, then not all pairs of the points can determine the same distance. Therefore it is natural to ask ([Er46]) at most how many times occurs the same distance. Sharp values were found only for $n \leq 14$.

Because just looking problem – despite numerous attempts – is still not fixed, it offers a wide variety of different approaches (often initiated by Erdős). It was examined many special cases and related problems:

what is the minimum number of distances determined by n points in the plane ([AEP91], [Bec83], [Chu84], [ChuST92], [Ele95], [ErF96], [ErF97]);

distribution of the distances and frequent long distances ([He56], [Ve85], [Ve87], [Ve96]);

distances of points on the sphere ([ChG85], [ChK73], [EHP89]) or distances between the vertices of convex polygons ([Al63], [Al72], [Fi95], [Fi97], [Se03]);

many other problems ([BáK01], [Fi98], [Jó09], [Koj01], [Koj02], [Wei12], [La*08], [Br98], [Br98b], [Er75], [Er82], [Er84], [Er86], [BáB94], [PuS10], [Zh11]).

Only few of those partial problems were completely solved.

* * *

One of the central problems of discrete geometry is so called decomposition, i.e. to divide the body into smaller parts. Very easily it can be shown that the circle with diameter D can be divided into three parts with smaller diameter, but it cannot be divided into two parts with smaller diameters. Easily can be proved also the first part of the 3-dimensional analog, namely that the ball of diameter D can be divided into four parts with smaller diameters.

Borsuk [Bors32] proved for any $d \geq 2$ that $d-$dimensional ball with diameter D cannot be divided into d parts with smaller diameters.

Theorem. ([Bors33]) Each region in E^2 with diameter D can be divided into three parts with smaller diameters.

In the proof of previous Borsuk's theorem the key role has the following (extremely useful) old geometric theorem.

Theorem. ([Pá21]) Every planar region F with radius D can be inscribed into the regular hexagon of the width D.

Borsuk's number is the smallest number $\beta(d)$ such that every set in E^d with radius 1 can be divided into $\beta(d)$ parts with the diameter less than 1. In a consequence of [Bors32] the inequality $\beta(d) \geq d+1$ is clear.

The conjecture $\beta(d) \leq d+1$ was supported by the extension of Borsuk's theorem for dimension 3, so that each region in E^3 with diameter D can be divided into four parts with smaller diameters (first proof was very complicated and can be found in [Eg55], much simplier proofs are e.g. in [Grü57], [He57]).

Counter-example found Kahn and Kalai [KaK93] in dimension 1326. Then arises a competition broke out in search of the smallest dimension where that conjecture is not valid: 946 - Nilli [Nil94], 561 - Raigorodskij [Rai97], 560 - Weissbach [We00], 323 - Hinrichs

[Hi02], 321 - Pikhurko [Pik02], 298 – Hinrichs and Richter [HiR03]. The authors of these improvements are agreed that the wanted smallest dimension is much smaller, perhaps somewhere between 4 and 10.

Conjecture. (Soifer [So10]) There is a bounded set in E^4, which cannot be divided into 5 parts with smaller diameters.

This is encouraging hypothesis, because dimension 4 is still quite clearly visible. Less encouraging is that since 2002, still holds the Hinrichs record 298.

* * *

All good luck and a lot of pleasant moments in solving open problems of this beautiful area of mathematics. Maybe it is sufficient to attempt the happiness by one coffee, because by Rényi *mathematician is a machine that converts coffee on theorems*.

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Selected Topics in the Extremal Graph Theory

Tomáš Vetrík*

Department of Mathematics and Applied Mathematics, University of the Free State, Bloemfontein, South Africa

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Abstract

Extremal problems in graph theory form a very wide research area. We study the following topics: the metric dimension of circulant graphs, the Wiener index of trees of given diameter, and the degree-diameter problem for Cayley graphs. All three topics are connected to the study of distances in graphs. We give a short survey on the topics and present several new results.

Keywords: Extremal graph theory, metric dimension, Wiener index, diameter, Cayley graph.

Classification: 05C35, 05C12

Introduction

Extremal graph theory is a sub-discipline of discrete mathematics focusing on identification and determination of maximal and minimal elements in partially ordered sets defined on, or arising from, specified families of graphs. Such type of analysis frequently reduces, in one way or another, to comparison of numerical parameters such as order of a graph (number of vertices), size (number of edges), diameter, degree, and so forth. More complex and important partial orders include graph minor relations, or containment as a subgraph (which may be required to be induced, topological, etc.). In fact, most results in the theory of finite graphs have been proved by considering extremal cases with respect to some partial order defined on graphs or on their parameter sets. In this sense, extremal graph theory plays a central role in the progress achieved in graph theory and, to some extent, also in the development of other branches of discrete mathematics.

We study the following problems:

- Metric dimension of circulant graphs,
- Wiener index of trees of given diameter,
- Degree-diameter problem for Cayley graphs.

Let G be a connected graph without loops and multiple edges. The distance d(u, v) between two vertices u, v in a graph G is the number of edges in a shortest path between them. The diameter of G is the greatest distance between all pairs of vertices of G. The degree of a vertex v is the number of edges incident to v. A tree is a graph, which does not contain cycles.

*Corresponding author: vetrikt@ufs.ac.za
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Metric dimension of circulant graphs

We consider the following problem. A robot moves in a space, which is modelled by a graph. The robot moves from a node to a node, and it can locate itself by the presence of distinctively labelled landmark nodes. The position of robot is represented by its distances to a set of landmarks. The problem is to find the minimum number of landmarks required, and to find out where they should be placed, such that the robot can always determine its location. In graph theory language, a minimum set of landmarks which uniquely determine the position of robot is called a metric basis, and the minimum number of landmarks is called the metric dimension.

A set of vertices W is a resolving set of a graph G if every two vertices of G have distinct representations of distances with respect to W. The number of vertices in a smallest resolving set is called the metric dimension and it is denoted by dim(G). A connected graph G has dim(G) = 1 if and only if G is a path. Cycles have metric dimension 2.

We study the metric dimension of circulant graphs. For $m \le n/2$ the circulant graph $C_n(1, 2, ..., m)$ consists of the vertices $v_0, v_1, ..., v_{n-1}$ and the edges $v_iv_{i+1}, v_iv_{i+2}, ..., v_iv_{i+m}$, where $0 \le i \le n-1$ and the indices are taken modulo n.

The metric dimension of circulant graphs has been extensively studied. Javaid, Rahim and Ali [6] showed that

$$\dim(C_n(1, 2)) = 3 \text{ if } n \equiv 0, 2, 3 \pmod 4, \text{ and } \dim(C_n(1, 2)) \le 4 \text{ if } n \equiv 1 \pmod 4.$$

Imran et. al. [5] proved that for any $n \ge 12$,

$$\dim(C_n(1, 2, 3)) = 4 \text{ if } n \equiv 2, 3, 4, 5 \pmod 6,$$

$$\dim(C_n(1, 2, 3)) \le 5 \text{ if } n \equiv 0 \pmod 6, \text{ and } \dim(C_n(1, 2, 3)) \le 6 \text{ if } n \equiv 1 \pmod 6.$$

We extend known results on the metric dimension of circulant graphs by presenting bounds on the metric dimension of circulant graphs with 4 generators.

Theorem 1. For any $n \ge 21$,

$$4 \le \dim(C_n(1, 2, 3, 4)) \le 6.$$

Finally we state two open problems. The first one is to find exact values of the metric dimension of $C_n(1, 2)$, $C_n(1, 2, 3)$ and $C_n(1, 2, 3, 4)$ for any n, and the second one is to find upper and lower bounds on $C_n(1, 2, ..., m)$ for $m \ge 5$.

Wiener index of trees of given diameter

The Wiener index is the oldest graph index. It has been investigated in the mathematical, chemical and computer science literature since the 1940's. With Wiener's discovery of a close correlation between the boiling points of certain alkanes and the sum of the distances between vertices in graphs representing their molecular structures, it became apparent that graph indices can potentially be used to predict properties of chemical compounds.

The Wiener index W(G) of a connected graph G is defined as the sum of the distances between all unordered pairs of vertices. The minimum value of the Wiener index of a graph (of a tree) of given order is attained by the complete graph (by the star), and the maximum value is attained by the path. It is not difficult to show that the extremal tree, which has the minimum Wiener index among trees of order n and diameter d, is the path of length d (containing d + 1 vertices) with the central vertex joined to the other n – d – 1 vertices; see [13].

The problem of finding an upper bound on the Wiener index of a tree (or graph) in terms of order and diameter is quite challenging; it was addressed by Plesník [11] in 1975, and restated by DeLaVina and Waller [3], but still remains unresolved to this date. We give a starting point to solving this long-standing problem. We present upper bounds on the Wiener index of trees of order n and diameter at most 6.

Theorem 2. Let T be a tree having n vertices and diameter d. Then the Wiener index of T is at most

(i) $5n^2/4 - 3n + 3$ if $d = 3$,

(ii) $2n^2 - 2n\sqrt{n-1} - 3n + 2\sqrt{n-1} + 1$ if $d = 4$,

(iii) $9n^2/4 - 2n^{3/2} + O(n)$ if $d = 5$,

(iv) $3n^2 - 2\sqrt{6}n^{3/2} - 2n + O(n^{1/2})$ if $d = 6$,

and the bounds are best possible.

The proof of this result can be found in [10]. The only tree of order n and diameter 2 is the star Sn having n – 1 leaves. Since any two leaves of the star are at distance 2, and the distance between the central vertex and any leaf is 1, the Wiener index of Sn is $n^2 - 2n + 1$. Let us mention that to find a sharp upper bound on the Wiener index for trees of given order and large diameter remains an open problem.

Note that there are indices which were introduced much later than the Wiener index, however upper bounds on these indices for trees of given order and diameter are known. For example, a sharp upper bound on the eccentric connectivity index of trees of given order and diameter was given in [9].

Degree-diameter problem for Cayley graphs

Suppose that one wants to set up a network in which each node has just a limited number of direct connections to other nodes, and one requires that any two nodes can communicate by a route of limited length. What is the maximum number of nodes one can have under the two constraints? It is clear that this question can be translated into the language of graph theory. The problem is to find the largest possible number of vertices in a graph of given maximum degree and diameter. Vertices of a graph represent nodes of a network, while edges represent connections.

A Cayley graph C(S, X) is specified by a group S and a unit-free generating set X for this group such that $X = X^{-1}$. The vertices of C(S, X) are the elements of S and there is an edge between two vertices u and v in C(S, X) if and only if there is a generator a in X such that v = ua.

The degree-diameter problem for Cayley graphs is to determine the largest number of vertices in a Cayley graph of given degree and diameter. Let $C_{d,k}$ be the largest order of a Cayley graph of degree d and diameter k. The number of vertices in a graph of maximum degree d and diameter k cannot exceed the Moore bound

$$M_{d,k} = 1 + d + d(d-1) + \ldots + d(d-1)^{k-1}.$$

In [1] Bannai and Ito improved the upper bound and showed that for any d, $k \geq 3$ there are no graphs of order greater than $M_{d,k} - 2$, therefore $C_{d,k} \leq M_{d,k} - 2$ for such d and k. Since the Moore graphs of diameter 2 and degree 3 or 7, and the potential Moore graph(s) of diameter 2 and degree 57 are non-Cayley (see [2]), Cayley graphs of order equal to the Moore bound exist only in the trivial cases when d = 2 or k = 1.

We focus on constructions of Cayley graphs of small diameter. Macbeth et al. [8] presented large Cayley graphs giving the bound $C_{d,k} \geq k((d-3)/3\}^k$ for any diameter $k \geq 3$ and degree $d \geq 5$. Let us also mention the Faber-Moore-Chen graphs [4] of odd degree $d \geq 5$, diameter k, such that $3 \leq k \leq (d+1)/2$, and order $((d+3)/2)! / ((d+3)/2 - k)!$. These graphs are vertex-transitive and in [8] it is proved that for any $k \geq 4$ and sufficiently large d the Faber-Moore-Chen graphs are not Cayley. Large Cayley graphs of given degree d and diameter k, where both d and k are small, were obtained by use of computers, see [7]. We state our results.

Theorem 3.

(i) $C_{d,3} \geq 3d^3/16$ for $d \geq 8$ such that d is a multiple of 4,

(ii) $C_{d,4} \geq 32(d/5)^4$ for $d \geq 10$ such that d is a multiple of 5,

(iii) $C_{d,5} \geq 25(d/4)^5$ for $d \geq 8$ such that d is a multiple of 4.

Proof. (i) Let H be a group of order $m \geq 2$ with unit element e. We denote by H^3 the product $H \times H \times H$. Let A be the automorphism of the group H^3, such that $A(x_1, x_2, x_3) = (x_3, x_1, x_2)$. We use the semidirect product S of H^3 and Z_{12} (the cyclic group with 12 elements) with multiplication given by

$$(x, y)(x', y') = (x\, A^y(x'), y + y'),$$

where A^y is the composition of A with itself y times, $x, x' \in H^3$ and $y, y' \in Z_{12}$. Elements of S will be written in the form $(x_1, x_2, x_3; y)$, where $x_1, x_2, x_3 \in H$ and $y \in Z_{12}$.

The generating set $X = \{ a_g, \bar{a}_{g'}, b_h, \bar{b}_{h'} \mid \text{for any } g, g', h, h' \in H\}$ where

$$a_g = (g, g, e; 1),\ \bar{a}_{g'} = (g', e, g'; -1),\ b_h = (h, e, e; 8) \text{ and } \bar{b}_{h'} = (e, h', e; 4).$$

It is easy to check that $X = X^{-1}$. The Cayley graph C(S, X) is of degree $d = |X| = 4m$ where $m \geq 2$ and order $|S| = 12m^3 = 12(d/4)^3 = 3d^3/16$.

We show that the diameter of C(S, X) is at most 3, which is equivalent to showing that each element of S can be obtained as a product of at most 3 generators of X. For any $x_1, x_2, x_3 \in H$ we have

$(x_1, x_2, x_3; 0) = (x_1, e, e; 8)(x_3, e, e; 8)(x_2, e, e; 8)$,

$(x_1, x_2, x_3; 1) = (x_1x_3^{-1}, e, e; 8)(x_3, x_3, e; 1)(e, x_2, e; 4)$,

$(x_1, x_2, x_3; 2) = (x_1, e, x_1; -1)(x_2, e, x_2; -1)(e, x_2^{-1}x_1^{-1}x_3, e; 4)$,

$(x_1, x_2, x_3; 3) = (x_1x_2^{-1}, e, e; 8)(x_3, e, e; 8)(x_2, e, x_2; -1)$,

$(x_1, x_2, x_3; 4) = (x_3, e, x_3; -1)(e, x_3^{-1}x_1x_2^{-1}, e; 4)(x_2, x_2, e; 1)$,

$(x_1, x_2, x_3; 5) = (x_1, e, e; 8)(x_3x_2^{-1}, e, e; 8)(x_2, x_2, e; 1)$,

$(x_1, x_2, x_3; 6) = (x_2x_3^{-1}, x_2x_3^{-1}, e; 1)(x_3, x_3, e; 1)(e, x_3x_2^{-1}x_1, e; 4)$.

It is easy to see that if $(x_1, x_2, x_3; y) = abc$, where $a, b, c \in X$ and $0 \le y \le 6$, then

$$(x_{y \pmod 3) + 1}^{-1}, x_{y + 1 \pmod 3) + 1}^{-1}, x_{y + 2 \pmod 3) + 1}^{-1}; -y) = c^{-1}b^{-1}a^{-1}.$$

Note that the diameter of C(S, X) cannot be smaller than 3, because the order of C(S, X) is greater than the Moore bound for diameter 2.

Proofs of (ii) and (iii) use similar techniques. □

By adding a few new elements to the generating sets, we get Cayley graphs of any degree $d \ge 10$ if $k = 4$, and any degree $d \ge 8$ if $k = 3$ or 5 (see [12]).

Corollary 1.

(i) $C_{d,3} \ge 3(d-3)^3/16$ for any $d \ge 8$,

(ii) $C_{d,4} \ge 32((d-8)/5)^4$ for any $d \ge 10$,

(iii) $C_{d,5} \ge 25((d-7)/4)^5$ for any $d \ge 8$.

These results improve the bounds of [8]. Particularly for diameter 3 we improve the lower bound considerably. It can be easily checked that the graphs of Faber, Moore and Chen are smaller than our graphs for diameter 3 and large degree, and they are larger than our graphs for diameters 4 and 5. However, for $k = 4$ and $d \ge 21$, and for $k = 5$ and $d \ge 23$, the Faber-Moore-Chen graphs are non-Cayley. To the best of our knowledge, for sufficiently large d there is no construction of Cayley graphs of degree d and diameter 3, 4 or 5 of order greater than the order of our graphs.

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Extensions of Cascades Created by Certain Function Systems

Jaroslav Beránek[a*] – Jan Chvalina[b]

[a] *Department of mathematics, Faculty of Education, Masaryk University, Poříčí 7, CZ - 603 00 Brno*
[b] *Department of mathematics, Faculty of Electrical Engineering and Communication, University of Technology, Technická 8, CZ - 616 00 Brno*

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Abstract

Considering three simple function systems which consist of power functions with odd exponents and two linear functions of one real variable, we are constructing actions of the additive group of all integers on the set of all real numbers, i.e. cascades. Using certain extensions based on prolongations of flows, we obtain the system of cascades which are mutually isomorphic. One from consequences of the result is that all solution sets of corresponding functional equations formed with the use of given functions are non-empty, moreover all solution sets consisting of permutations of the set of all reals are non-empty, as well.

Keywords: Mono–unary algebra, orbital decomposition, cascade.

Classification: 26A09, 37B05

Introduction

The theory of dynamical systems is a very broad mathematical area historically arising from the theory of ordinary differential equations. As it has been mentioned in [1, 2], the qualitative theory of ordinary differential equations and the theory of dynamical systems arose within the theory of differential equations; in time, the theory of dynamical systems attained a definite autonomy, and it can now be regarded as an independent branch of mathematics, which continues to develop intensively. It retains a close connection with the theory of differential equations, and the boundary between them is not particularly sharp. At the same time, the theory of dynamical systems has established new connections with the branches of mathematics which appear ever more essential for certain questions in the theory of dynamical systems. Even the concept of a dynamical system has itself evolved considerably – [2, 3, 4, 7, 12].

In this contribution we will concentrate ourselves onto cascades (called also discrete flows). Let us recall that a flow is in fact a one-parameter group or semigroup of transformations acting on a set M, which is called the phase space of the flow. In other words, associated to each $t \in \boldsymbol{R}$ (the set of all real numbers) or $t \in \boldsymbol{R_0^+}$ (the set of all non-negative real numbers), there is a mapping $g^t : M \to M$ such that the group property holds, i. e.

$$g_0 = id_M,\ g^{t+s} = g^t \circ g^s,$$

*Corresponding author: beranek@ped.muni.cz
DOI: 10.17846/AMN.2015.1.1.50-56

for all *t, s* under consideration. A cascade differs from a flow in that the maps g^t are only defined for $t \in \boldsymbol{Z}$ (the set of all integers) or $t \in \boldsymbol{N_0}$ (the set of all non-negative integers).

If $k \in \mathbf{Z}$, the notation g^k denotes the k-th iterate of the mapping $g = g^1$ for $k > 0$ and the k-th iterate of g^{-1} when $k < 0$. The name „cascade“ is used to contrast it to a „flow“. Flows are most often encountered in applications, but cascades also appear. For example, in ecology one might want to study changes in a population with non-overlapping adult generations. Here the generations play the role of discrete time. Nevertheless, the main significance of cascades lies in the fact that they are usually technically somewhat simpler than flows; at the same time, the essence of the matter may be the same in both cases. Thus, results obtained for cascades frequently carry over to flows, often not by way of a formal reduction, but by some modifications of the proofs.

Preliminaries

Let us recall that a cascade is sometimes called a discrete flow or a flow with a discrete time. Let us mention other necessary concepts. Functions $f: \mathbf{R} \to \mathbf{R}$, $g: \mathbf{R} \to \mathbf{R}$ are called conjugated if there is a bijection $h: \mathbf{R} \to \mathbf{R}$ such that $h \circ f = g \circ h$ [10]. Corresponding mono-unary algebras $(\mathbf{R},f)$, $(\mathbf{R},g)$ are then said to be isomorphic and the mapping h is an isomorphism. In general, a mapping $h: (\mathbf{R}, f) \to (\mathbf{R}, g)$ with the above property $(h(f(x)) = g(h(x)), x \in \mathbf{R})$, is called a homomorphism of the mono-unary algebra $(\mathbf{R}, f)$ into the mono-unary algebra $(\mathbf{R},g)$ - [9].

Let $(G, \cdot, e)$ be a group with the unit e, $X \neq \phi$, $\Theta: X \times G \to X$ be a mapping satisfying these conditions:

(i) $\Theta(x, e) = x$ for any $x \in X$ (the Identity axiom),

(ii) $\Theta(\Theta(x, a), b) = \Theta(x, a \cdot b)$ for all $x \in X$, $a, b \in G$ (the Homomorphism axiom or the Mixed associability condition – MAC).

Then the triad $\mathbf{A} = (X, G, \Theta)$ is said to be the action of the group G on the set X [6] or a discrete dynamical system with the phase group G and the phase set (space) X [12]. If $G = \mathbf{R}$, X is a metric space or a topological space, $\Theta: X \times G \to X$ is a continuous mapping, then the action $\mathbf{A}$ is called a flow. If $G = (\mathbf{Z}, +)$ and X is a set without – not necessary – any additional structure, the action $\mathbf{A}$ is termed a cascade.

If $\mathbf{A} = (X, G, \Theta_A)$, $\mathbf{B} = (Y, G, \Theta_B)$ are cascades with the same phase group G, then a mapping $h: X \to Y$ such that $\Theta_B(h(x),a) = h(\Theta_A(x,a))$ for any pair $[x,a] \in X \times G$ is said to be a homomorphism of the cascade $\mathbf{A}$ into the cascade $\mathbf{B}$. If h is a bijection, then this homomorphism is called an isomorphism. If some isomorphism $h: \mathbf{A} \to \mathbf{B}$ exists, the cascades $\mathbf{A}$, $\mathbf{B}$ are said to be isomorphic and we write $\mathbf{A} \cong \mathbf{B}$.

Let (X, f) be a mono-unary algebra. Let us denote by $f^n: X \to X$ the n-th iteration of the mapping f. Let us suppose $f: X \to X$ is a bijective mapping. To the algebra (X, f) we can assign the action $\mathbf{A_f} = (\mathbf{Z}, X, \Theta_f)$, where $\Theta_f(m, x) = f^m(x)$ for any $m \in \mathbf{Z}$ and $x \in X$, where f^m for $m = -n$, $n \in \mathbf{N}$ is defined $f^m = (f^{-1})^n$ because the inverse bijection f^{-1} to the bijection f exists. It is evident that the above conditions (i), (ii) are satisfied, thus the action $\mathbf{A_f}$ is a cascade. Moreover, if (X,f), (Y,g) are mono-unary algebras and $\mathbf{A_f} = (\mathbf{Z}, X, \Theta_f)$, $\mathbf{A_g} = (\mathbf{Z}, Y, \Theta_g)$ are corresponding cascades, then for any homomorphism $h: (X, f) \to (Y, g)$ (i. e. $h(f(x)) = g(h(x))$, $x \in X$) we have that $h: \mathbf{A_f} \to \mathbf{A}_g$ is a homomorphism of cascades, thus the described construction is functorial.

Orbital decompositions of mono-unary algebras

Let us remind the concept of an orbital decomposition of a mono-unary algebra [9, 11]. Let us suppose (X, f) is a mono-unary algebra where the mapping $f: X \to X$ is not necessary bijective or injective. We define a binary relation $\sim_f \subset X \times X$ in this way: For $x, y \in X$ we put $x \sim_f y$ whenever there exists a pair $[m, n] \in \mathbf{N}_0 \times \mathbf{N}_0$ such that $f^n(x) = f^m(y)$. The relation $\sim_f$ is an equivalence on X called a KW-equivalence (Kuratowski-Whyburn). Blocks $S \in X/\sim_f$ are called f-orbits. A subalgebra $(S, f/S)$ (here symbols f/S mean the restriction of the function f onto the set S) of the mono-unary algebra (X, f) is said to be a component of the algebra (X, f) (of course, there holds $f(S) \subset S$). If $\{(S_\alpha, f_\alpha); \alpha \in I\}$ is the system of all components of the algebra (X, f) (here $f_\alpha = f / S_\alpha$) , we write

$$(X, f) = \sum_{\alpha \in I} (S_\alpha, f_\alpha)$$

and this sum is termed as an orbital decomposition of the mono-unary algebra (X, f).

Now let us consider three one-parametrical systems of elementary functions:

$$\varphi_k(x) = x^{2k+1},\ \psi_k(x) = (k+1)x,\ \xi_k(x) = x + k,\ x \in \mathbf{R},\ k \in \mathbf{N}.$$

All considered functions are bijections of the set $\mathbf{R}$ of all real numbers onto itself; functions φ_k have three fixed points $-1, 0, 1$ for any $k \in \mathbf{N}$, functions ψ_k have exactly one fixed point 0, functions ξ_k do not have any fixed point. Denoting by $(0, 1)$ the open interval $\{x \in \mathbf{R}; 0 < x < 1\}$, we obtain the following assertion describing orbital decompositions of mono-unary algebras $(\mathbf{R}, \varphi_k)$, $(\mathbf{R}, \psi_k)$, $(\mathbf{R}, \xi_k)$.

Lemma: The mono-unary algebras $(\mathbf{R}, \varphi_k)$, $(\mathbf{R}, \psi_k)$, $(\mathbf{R}, \xi_k)$ for $k \in \mathbf{N}$ have these orbital decompositions:

$$(\mathbf{R}, \varphi_k) = (\{-1\}, id) + (\{0\}, id) + (\{1\}, id) + \sum_{\alpha \in (0,1)} (X_\alpha, \varphi_{\alpha,k}),$$

$$(\mathbf{R}, \psi_k) = (\{0\}, id) + \sum_{\alpha \in (0,1)} (Y_\alpha, \psi_{\alpha,k}), \quad (\mathbf{R}, \xi_k) = \sum_{\alpha \in (0,1)} (K_\alpha, \xi_{\alpha,k}),$$

where $\varphi_{\alpha,k} = \varphi_k / X_\alpha$, $\psi_{\alpha,k} = \psi_k / Y_\alpha$, $\xi_{\alpha,k} = \xi_k / K_\alpha$, $\alpha \in (0, 1)$ and

$$(X_\alpha, \varphi_{\alpha,k}) \cong (Y_\alpha, \psi_{\alpha,k}) \cong (K_\alpha, \xi_{\alpha,k}) \cong (\mathbf{Z}, \nu_{\mathbf{Z}})$$

for any index $\alpha \in (0, 1)$ and $\nu_{\mathbf{Z}}(m) = m + 1$, $m \in \mathbf{Z}$. □

If (X, f), (Y, g) are mono-unary algebras, let us denote by $Hom((X, f), (Y, g))$ the set of all homomorphisms of the algebra (X, f) into the algebra (Y, g), and similarly, if $\mathbf{A}$, $\mathbf{B}$ are cascades, then $Hom(\mathbf{A}, \mathbf{B})$ means the set of all homomorphisms of the cascade $\mathbf{A}$ into the cascade $\mathbf{B}$.

Constructions of homomorphisms of mono-unary algebras are described in [9], where various modifications and applications of presented constructions are also included. For example, all homomorphisms of the algebra $(\mathbf{R}, \varphi_k)$ into the algebra $(\mathbf{R}, \psi_k)$ can be obtained in this way: Let T be the set of all functions $\tau : (0, 1) \to (0, 1)$ and Λ_α be the set of all isomorphisms

$$\lambda : (X_\alpha, \varphi_{\alpha,k}) \to (Y_{\tau(\alpha)}, \psi_{\tau(\alpha),k})$$

if $\tau(\alpha) \neq 0$ including the constant mapping $\lambda : X_\alpha \to \{0\}$ for $\alpha \in (0, 1)$. Now we put $f(-1) = f(0) = f(1) = 0$, and if $x \in X_\alpha$, we define $f(x) = \lambda(x)$ for a concrete function $\lambda \in \Lambda_\alpha$ and $\tau \in T$. Then $f \in Hom((\mathbf{R}, \varphi_k), (\mathbf{R}, \psi_k))$ and if the function τ is running over the set T and λ is running over the set Λ_α, $\alpha \in (0, 1)$, we obtain all functions $f \in Hom((\mathbf{R}, \varphi_k), (\mathbf{R}, \psi_k))$ – [9].

Main results

Let us denote by $\mathbf{A}(\varphi_k)$, $\mathbf{A}(\psi_k)$, $\mathbf{A}(\xi_k)$ cascades assigned by the above presented functorial construction to mono-unary algebras $(\mathbf{R}, \varphi_k)$, $(\mathbf{R}, \psi_k)$, $(\mathbf{R}, \xi_k)$ in the given order. Considering the fact that $card\ T = c^c$, where $c = exp\,\aleph_0$, and that for any function $h : \mathbf{R} \to \mathbf{R}$ there holds

$$\xi_k(h(0)) = h(0) + k \neq h(0) = h(\varphi_k(0)),\ k \in \mathbf{N},$$

and similarly

$$\xi_k(h(0)) = h(0) + k \neq h(0) = h(\psi_k(0)),\ k \in \mathbf{N}$$

we obtain with respect to the above lemma and to the mentioned construction of sets of homomorphisms of mono-unary algebras in question the following assertion.

Proposition: For any $k \in \mathbf{N}$ we have

$$card\ Hom(\mathbf{A}(\varphi_k), \mathbf{A}(\psi_k)) = card\ Hom(\mathbf{A}(\psi_k), \mathbf{A}(\varphi_k)) =$$
$$card\ Hom(\mathbf{A}(\xi_k), \mathbf{A}(\varphi_k)) = card\ Hom(\mathbf{A}(\xi_k), \mathbf{A}(\psi_k)) = c^c,$$
$$Hom(\mathbf{A}(\varphi_k), \mathbf{A}(\xi_k)) = Hom(\mathbf{A}(\psi_k), \mathbf{A}(\xi_k)) = \phi.$$ □

The asymmetry expressed in the Proposition can be deleted by the below described construction of extensions.

Definition: Let $\mathbf{A}$, $\mathbf{B}$ be different cascades such that there exists an injective homomorphism (i.e. an embedding) $h: \mathbf{A} \to \mathbf{B}$. Then the cascade $\mathbf{B}$ is said to be an extension of the cascade $\mathbf{A}$.

Theorem: Let us suppose $k \in \mathbf{N}$. There exist extensions $\tilde{\mathbf{A}}(\tilde{\psi}_k)$, $\hat{\mathbf{A}}(\hat{\xi}_k)$ of the cascades $\mathbf{A}(\psi_k)$, $\mathbf{A}(\xi_k)$ such that

$$\mathbf{A}(\varphi_k) \cong \tilde{\mathbf{A}}(\tilde{\psi}_k) \cong \hat{\mathbf{A}}(\hat{\xi}_k).$$

Proof: The extension of the cascade $\mathbf{A}(\psi_k)$ consists in an adding of two different critical points (called also equilibrium points) to its phase set $\mathbf{R}$ and also in the extension of the action Θ_ψ onto the new set of three critical points.

Let us denote $\tilde{\mathbf{R}} = \mathbf{R} \cup \{-\infty, \infty\}$, where $-\infty, \infty$ are elements which do not belong to the set $\mathbf{R}$. In terms of operations on the class of ordered sets the chain $(\tilde{\mathbf{R}}, \leq)$ can be expressed as the ordinal sum

$$(\tilde{\mathbf{R}}, \leq) = \{-\infty\} \oplus (\mathbf{R}, \leq) \oplus \{\infty\}.$$

Since

$$\lim_{x\to-\infty} \psi_k(x) = \lim_{x\to-\infty} (k+1)x = -\infty \text{ for any } k \in \mathbf{N}$$

and

$$\lim_{x\to\infty} \psi_k(x) = \lim_{x\to\infty} (k+1)x = \infty \text{ for any } k \in \mathbf{N}$$

we define $\tilde{\psi}_k(-\infty) = -\infty$, $\tilde{\psi}_k(\infty) = \infty$ and $\tilde{\psi}_k(x) = \psi_k(x)$ for each $x \in \boldsymbol{R}$. Now we have

$$(\tilde{\boldsymbol{R}}, \tilde{\psi}_k) = (\{-\infty\}, id) + (\{0\}, id) + (\infty\}, id) + \sum_{\alpha \in (0,1)} (X_\alpha, \psi_{\alpha,k}),$$

Where – as above $-(X_\alpha, \psi_{\alpha,k}) \cong (\boldsymbol{Z}, \nu_{\boldsymbol{Z}})$ for any $\alpha \in (0, 1)$, thus $(\tilde{\boldsymbol{R}}, \tilde{\psi}_k) \cong (\boldsymbol{R}, \varphi)$, $k \in \boldsymbol{N}$. Then denoting

$$\tilde{\boldsymbol{A}}(\tilde{\psi}_k) = (\tilde{\boldsymbol{R}}, \boldsymbol{Z}, \tilde{\Theta}_{\tilde{\psi}}),$$

where $\tilde{\Theta}_{\tilde{\psi}}(x, m) = (\tilde{\psi})^m(x)$, $x \in \tilde{\boldsymbol{R}}$ and $m \in \boldsymbol{Z}$, we have $\tilde{\boldsymbol{A}}(\tilde{\psi}_k) \cong \boldsymbol{A}(\varphi_k)$ for any $k \in \boldsymbol{N}$. Since for $h(x) = x$, $x \in \boldsymbol{R}$

$$h(\Theta_\psi(x, m)) = \Theta_\psi(x, m) = \Theta_\psi(h(x), m) = \tilde{\Theta}_{\tilde{\psi}}(h(x), m)$$

for each pair $[x, m] \in \boldsymbol{R} \times \boldsymbol{Z}$, the cascade $\tilde{\boldsymbol{A}}(\tilde{\psi}_k)$ is an extension of the cascade $\boldsymbol{A}(\psi_k)$ (here the injection $h: \boldsymbol{R} \to \tilde{\boldsymbol{R}}$ is the corresponding embedding of $\boldsymbol{A}(\psi_k)$ into $\tilde{\boldsymbol{A}}(\tilde{\psi}_k)$ for each $k \in \boldsymbol{N}$.

Mono-unary algebras $(\boldsymbol{R}, \xi_k)$, $k \in \boldsymbol{N}$ do not possess any critical point, hence we add three critical points, namely $x_1 = -\infty$, $x_2 = \infty$ and $x_3 = C_{0,k}$, where $C_{0,k}$ for a given k is the class of the decomposition $\boldsymbol{Z}$ modulo k which forms zero of the group $\boldsymbol{Z}$ / *mod k,* i.e.

$$C_{0,k} = \{\dots, -3k, -2k, -k, 0, k, 2k, 3k, \dots, nk, \dots\}.$$

Similarly as above

$$\lim_{x \to -\infty} \xi_k(x) = -\infty, \ \lim_{x \to \infty} \xi_k(x) = \infty, \ k \in \boldsymbol{N}$$

and moreover

$$\xi_k(C_{0,k}) = C_{0,k} + k = \{z + k;\ z \in C_{0,k}\}$$

for an arbitrary $k \in \boldsymbol{N}$. We define

$$\hat{\boldsymbol{R}}_{\boldsymbol{k}} = \boldsymbol{R} \cup \{-\infty, \infty, C_{0,k}\}, \text{ i.e. } (\hat{\boldsymbol{R}}_{\boldsymbol{k}}, \leq) = \{C_{0,k}\} + (\tilde{\boldsymbol{R}}, \leq)$$

and $\hat{\xi}_k(-\infty) = -\infty$, $\hat{\xi}_k(\infty) = \infty$, $\hat{\xi}_k(C_{0,k}) = C_{0,k}$, $\hat{\xi}_k(x) = \xi_k(x) = x + k$ for any $x \in \boldsymbol{R}$ and $k \in \boldsymbol{N}$.

Then evidently

$$(\boldsymbol{R}, \varphi_k) \cong (\tilde{\boldsymbol{R}}, \tilde{\psi}_k) \cong (\hat{\boldsymbol{R}}_{\boldsymbol{k}}, \hat{\xi}_k) \tag{1}$$

for any $k \in \boldsymbol{N}$. Defining functions $\hat{\Theta}_{k,\hat{\xi}} : \hat{\boldsymbol{R}}_{\boldsymbol{k}} \times \boldsymbol{Z} \to \hat{\boldsymbol{R}}_{\boldsymbol{k}}$ by the formula $\hat{\Theta}_{k,\hat{\xi}}(x, m) = (\hat{\xi}_k)^m(x)$ for any $x \in \hat{\boldsymbol{R}}_{\boldsymbol{k}}$ and any $m \in \boldsymbol{Z}$, we obtain, similarly as above, that cascades $\hat{\boldsymbol{A}}_{\boldsymbol{k}} = (\hat{\boldsymbol{R}}_{\boldsymbol{k}}, \boldsymbol{Z}, \hat{\Theta}_{k,\hat{\xi}})$ are extensions of the cascades $\boldsymbol{A}(\xi_k)$ for any $k \in \boldsymbol{N}$. Consequently, with respect to (1) we have for each $k \in \boldsymbol{N}$

$$\boldsymbol{A}(\varphi_k) \cong \tilde{\boldsymbol{A}}(\tilde{\psi}_k) \cong \hat{\boldsymbol{A}}(\hat{\xi}_k). \qquad \square$$

Concluding remarks

Remark 1: From the just proved theorem there follows immediately that all functional equations

$$f(\rho(x)) = \sigma f(x), \text{ for } \rho, \sigma \in \{\varphi_k, \tilde{\psi}_k, \hat{\xi}_k\}$$

have solution sets of the cardinality c^c and moreover all of the mentioned functional equations have infinitely many bijective solutions.

Remark 2: Treated structures are very special objects belonging to the theory of dynamical systems or to dynamical topology. Bibliography for these topics up to 1972 was compiled by Walter Helbig Gottschalk, Wesleyan University, Middletown, Connecticut. This Bibliography contains 178 pages and lists 2475 numbered items – cf. [7], p. 164 or [4], p. 301. Further, discrete dynamical systems with input discrete groups or semigroups constitute in fact a certain class of automata without outputs also called quasi-automata – [5, 8]. Literature devoted to these structures belonging into the algebraic theory of automata is very comprehensive. Let us notice in this connection that the actions of groups from the purely algebraic point of view are treated in chapter 5 of the monography [6] and in the paper [8].

Remark 3: In the theory of dynamical systems, in particular in the theory of continuous flows – [2] there are investigated some concepts and properties of objects which can be transferred onto cascades. Chapter II, [2] of the monography [2] contains definitions of invariant, positively invariant and negatively invariant subsets of a phase set of a flow. The mentioned notions can be transferred onto cascades without any formal change. Also theorems 1. 3. through 1. 5., pp. 12-13 [2] are valid. In particular, theorem 1. 5. says that a set $M \subset X$ (the phase set) is invariant (i. e. $\Theta(x, p) \in M$ for all $x \in M$ and all $p \in \boldsymbol{Z}$ in our case) if and only if it is both positively invariant ($\Theta(x, p) \in M$ for all $x \in M$ and all $p \in \boldsymbol{Z}^+$) and negatively invariant (similarly as above, but $p \in \boldsymbol{Z}^-$). Since a set $M \subset \boldsymbol{R}$ is invariant in a cascade $\boldsymbol{A} = (\boldsymbol{R}, \boldsymbol{Z}, \Theta)$ if and only if M is the union of a system of orbits, it is clear that theorem 1. 5. [2] is also true in our case. Also the concept of a trajectory $\gamma(x)$ [2] (or a semi-trajectory) in our case is a single orbit (or an iterated sequence – i. e. a splinter [10]) containing the number $x \in \boldsymbol{R}$. Terms of the first positive prolongation and of the first prolongational limit set are near to our construction of extensions of cascades. Thus, all the mentioned concepts can be illustrated by our considered structures.

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Pencils of Planes and Spheres in Problem Solving

Martin Billich*

Department of mathematics, Faculty of Education, Catholic University in Ružomberok.
Hrabovská cesta 1, SK - 034 01 Ružomberok

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Abstract

In the present paper we present some facts in the theory of a pencil of planes and spheres which can be defined as a linear combination of their equations. In addition, we discuss possible approach how to use the pencils of planes or pencil of spheres in solving some problems in the analytic geometry.

Keywords: Pencil of planes, pencil of spheres, linear combination.

Classification: G44, D44

Introduction

The notion of linear combination in problem solving within analytic geometry is not a common one. Most often, we use a linear combination as related to a pencil of lines or of circles in Euclidean plane (see [1], [2]). Like in the case of pencils of planar curves, a notion of a pencil of planes and spheres is very useful, and we define it as follows. A pencil of planes or of spheres is the family of all planes or of spheres in space which pass through the intersection of two fixed planes or the intersection of a sphere and a plane or two spheres. In other words, a pencil of planes (spheres) is the family of planes (spheres) through a fixed line (circle). In this article, we look at some pencils in three dimensional Euclidean space, and we demonstrate a method in which pencils are used as tools in problem solving.

Pencil of planes

Let consider two planes P_1 and P_2 having a common line of intersection, given by their equations:

$$P_1 : a_1x + b_1y + c_1z + d_1 = 0 \tag{1}$$

$$P_2 : a_2x + b_2y + c_2z + d_2 = 0 \tag{2}$$

If we multiply an equation (1) by α and (2) by β, where α and β are real parameters not both equal to zero, and we add both equations, we have

$$\alpha(a_1x + b_1y + c_1z + d_1) + \beta(a_2x + b_2y + c_2z + d_2) = 0 \tag{3}$$

i.e. we have a *linear combination* of equations (1) and (2). From the condition of existence of the non-empty intersection of planes P_1 and P_2 follows a linearly independence of the normal vectors (a_1, b_1, c_1), and (a_2, b_2, c_2) to these planes. The set of all points of 3-

*Corresponding author: billich@ku.sk
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dimensional Euclidean space the coordinates of which satisfy equation (3) is a plane that contains common points of plane P_1 and P_2, because their coordinates satisfy equations (1) and (2) simultaneously. In other words, linear combination (3) is the equation of pencil of planes that contain the unique intersection line of P_1 and P_2.

Problem 1. Consider the line L of intersection of the two planes given by

$$x + y - 3 = 0, \quad x - 2y + z = 0.$$

Also, consider the plane $P: 2x + z + 1 = 0$ (Figure 1). Find an equation of plane containing the line L and perpendicular to the plane P.

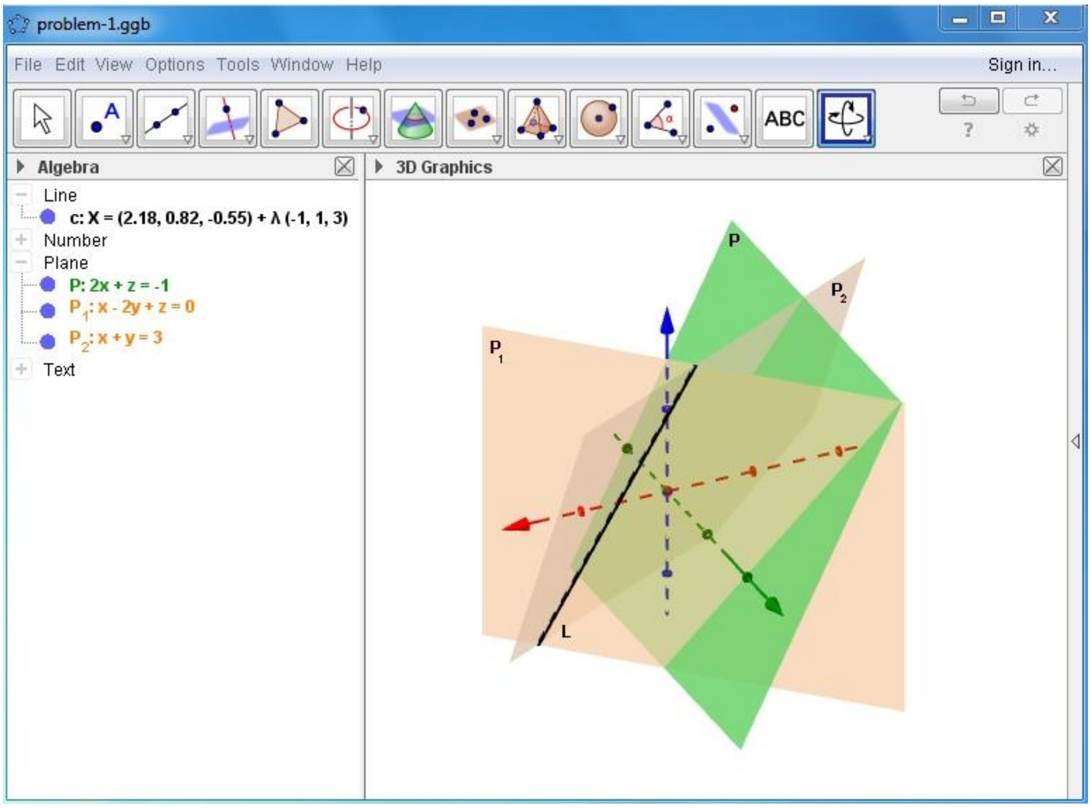

Figure 1

Solution. The plane is determined by equation (3) in the form

$$\alpha(x + y - 3) + \beta(x - 2y + z) = 0 \text{ or } (\alpha + \beta)x + (\alpha - 2\beta)y + \beta z - 3\alpha = 0.$$

The normal vector to the plane is perpendicular to the plane P, hence their dot product is

$$2.(\alpha + \beta) + 0.(\alpha - 2\beta) + 1.\beta = 0.$$

This is equivalent to $2\alpha = -3\beta$. If we set $\alpha = 3$, then $\beta = -2$, and the plane we are looking for has equation

$$x + 7y - 2z - 9 = 0.$$

Pencil of spheres

Let us consider the plane P, and the sphere S defined by the following equations:

$$P: ax + by + cz + d = 0 \tag{4}$$

$$S:(x-c_1)^2+(y-c_2)^2+(z-c_3)^2-r^2=0 \tag{5}$$

If we multiply an equation (4) by a real number α and sum both equations, we have

$$\alpha(ax+by+cz+d)+[(x-c_1)^2+(y-c_2)^2+(z-c_3)^2-r^2]=0 \tag{6}$$

This is equivalent to the following equation

$$(x-A)^2+(y-B)^2+(z-C)^2=A^2+B^2+C^2-D \tag{7}$$

where

$$A=c_1-\frac{\alpha}{2}a,\ B=c_2-\frac{\alpha}{2}b,\ C=c_3-\frac{\alpha}{2}c,\ D=c_1^2+c_2^2+c_3^2-r^2.$$

What can we say about the set of all points in three dimensional Euclidean space, which coordinates satisfy (7)?

If $A^2+B^2+C^2-D<0$ then there doesn't exist a point the coordinates of which satisfy equation (6). Otherwise it can be a point or a sphere. If the sphere S and the plane P have at least two common points then (7) is the equation of a sphere which contains all of their common points, because their coordinates satisfy the equations (4) and (5).

Problem 2. Let C be a circle given as the intersection of plane P and sphere S with the equations

$$P: x-y-z+2=0,\quad S:(x-1)^2+(y+2)^2+z^2-16=0.$$

Find the equation of a sphere which contains the circle C and passing through the point $M[2,-3,2]$.

Solution. The equation of our sphere has the form

$$\alpha(x-y-z+2)+[(x-1)^2+(y+2)^2+z^2-16]=0$$

As it contains the point M, its coordinates $[2,-3,2]$ satisfy the above equation, and we have $\alpha=2,$ and the sphere we are looking for has equation

$$x^2+(y+1)^2+(z-1)^2=9.$$

Problem 3. Let C be a circle given as the intersection of plane P and sphere S with the equations

$$P: x-y-z+2=0,\quad S:(x-1)^2+(y+2)^2+z^2-18=0.$$

Also, consider the line $L: y-3=0, 2y+z-5=0.$ Find the equation of a sphere which contains the circle C and tangent to the line $L.$

Solution. The equation of the sphere in question has the form

$$\alpha(x-y-z+2)+[(x-1)^2+(y+2)^2+z^2-18]=0.$$

Every point on the line L has coordinates $[t,3,-1], t\in R.$ Substituting these coordinates into the above equation, we have the quadratic equation with parameter α in the form

$$t^2+(\alpha-2)t+9=0.$$

The solution of this equation are x-coordinates of the common points of our sphere and given line L. Because L is the line tangent to the sphere, discriminant of the quadratic equation is equal to zero, i.e. $(\alpha-2)^2-4.1.9=0$. From this it is obvious that $\alpha=8$, or $\alpha=-4$. Therefore, there are two such spheres we are looking for (Figure 2):

$$(x+3)^2+(y-2)^2+(z-4)^2=26 \text{ and } (x-3)^2+(y+4)^2+(z+2)^2=50.$$

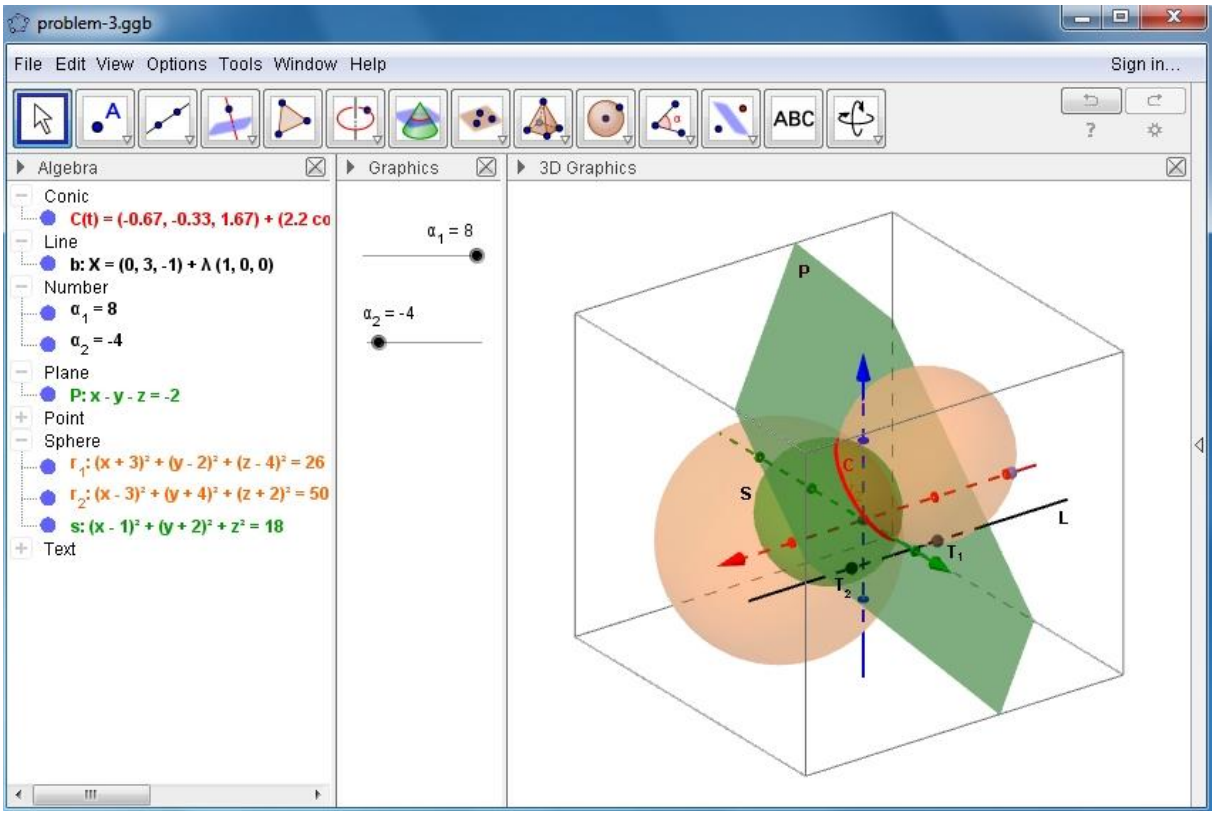

Figure 2

Now, we consider two non-concentric spheres with the equations:

$$S_1:(x-m_1)^2+(y-m_2)^2+(z-m_3)^2-r_1^2=0 \tag{8}$$

$$S_2:(x-n_1)^2+(y-n_2)^2+(z-n_3)^2-r_2^2=0 \tag{9}$$

If we multiply an equation (8) by α and (9) by β, where α and β are real parameters not both equal to zero, and if we add both equations, we obtain

$$(\alpha+\beta)(x^2+y^2+z^2)-2[(\alpha m_1+\beta n_1)x+(\alpha m_2+\beta n_2)y+(\alpha m_3+\beta n_3)z]+$$

$$\alpha m_1^2+\beta n_1^2+\alpha m_2^2+\beta n_2^2+\alpha m_3^2+\beta n_3^2-\alpha r_1^2-\beta r_2^2=0 \tag{10}$$

What can we say about the set of all points in space, the coordinates of which satisfy (10)?

If $\alpha+\beta=0$, then without loss of generality, we can set $\alpha=1$, $\beta=-1$, and (10) is the equation of the plane

$$2(n_1-m_1)x+2(n_2-m_2)y+2(n_3-m_3)z+(m_1^2+m_2^2+m_3^2-r_1^2)-(n_1^2+n_2^2+n_3^2-r_2^2)=0 \tag{11}$$

Given spheres are non-concentric, therefore at least one of the coefficients $(m_i-n_i), i=1,2,3$ is not equal to zero. This plane is perpendicular to the line of centers of both given spheres.

It will be called the *radical plane* of all the spheres of pencil determined by given spheres S_1 and S_2.

If $\alpha+\beta\neq 0,$ then we could write equation (10) in the form

$$(x-A)^2+(y-B)^2+(z-C)^2=A^2+B^2+C^2-D \qquad (12)$$

where

$$A=\frac{\alpha m_1+\beta n_1}{\alpha+\beta},\ B=\frac{\alpha m_2+\beta n_2}{\alpha+\beta},\ C=\frac{\alpha m_3+\beta n_3}{\alpha+\beta},\ D=\frac{\alpha(m_1^2+m_2^2+m_3^2)+\beta(n_1^2+n_2^2+n_3^2)}{\alpha+\beta}$$

A number of the common points of these spheres depends on the right side of equation (12). Denote $R=A^2+B^2+C^2-D.$

(i) If $R<0,$ then there doesn't exist a point satisfying (12);

(ii) If $R=0,$ then only the point $[A,B,C]$ satisfies (12);

(iii) If $R>0,$ then equation (12) determines the sphere with center $[A,B,C]$ and radius $\sqrt{R}.$

The *pencil* determined by spheres S_1 and S_2 is the set of all spheres with equations of the form (10). If the spheres intersect at least in two points, the radical plane contains their circle of intersection. A distance from the center $M[m_1,m_2,m_3]$ of sphere S_1 to the radical plane is given by

$$d=\frac{|2(n_1-m_1)m_1+2(n_2-m_2)m_2+2(n_3-m_3)m_3+(m_1^2+m_2^2+m_3^2-r_1^2)-(n_1^2+n_2^2+n_3^2-r_2^2)|}{\sqrt{[2(n_1-m_1)]^2+[2(n_2-m_2)]^2+[2(n_3-m_3)]^2}}$$

which simplifies to

$$d=\frac{|(n_1-m_1)^2+(n_2-m_2)^2+(n_3-m_3)^2+r_1^2-r_2^2)|}{2\sqrt{(n_1-m_1)^2+(n_2-m_2)^2+(n_3-m_3)^2}}.$$

a) If $d>r_1,$ then the intersection of given spheres be an empty set;

b) If $d=r_1,$ then the intersection of given spheres be only one point P (as a point of intersection of the line through the centeres of the given spheres and the radical plane);

c) If $d<r_1,$ then the intersection of given spheres be a circle in the radical plane, with radius $\sqrt{r_1^2-d^2}$ and the center $P.$

Problem 4. Determine the equation of a sphere through the circle of intersection of two given spheres S_1 and S_2 and tangent to the plane $P,$ where

$$S_1: x^2+y^2+z^2=9,\ \ S_2:(x-1)^2+(y+1)^2+(z-2)^2=7,\ \ P: x-5=0.$$

Solution. The circle we are looking for has the equation

$$\alpha(x^2+y^2+z^2-9)+\beta[(x-1)^2+(y+1)^2+(z-2)^2-7]=0,\ \ \alpha+\beta\neq 0.$$

By substituting $x=5$ (from the equation of P) into this equation we have

$$(\alpha+\beta)y^2+2\beta y+(\alpha+\beta)z^2-4\beta z+16\alpha+14\beta=0.$$

which simplifies to

$$(y-A)^2+(z-B)^2=A^2+B^2-C, \text{ and } A=\frac{\beta}{\alpha+\beta},\ B=\frac{2\beta}{\alpha+\beta},\ C=\frac{16\alpha+14\beta}{\alpha+\beta}.$$

As the sphere to be found should be tangent to the plane P, the right side A^2+B^2-C of above equation will be equal to zero. We get

$$5\beta^2=(\alpha+\beta)(16\alpha+14\beta) \quad\Rightarrow\quad \left(\frac{8\alpha}{\beta}+3\right)\left(\frac{2\alpha}{\beta}+3\right)=0.$$

If we set $\alpha=-3$ or $\alpha=3$ then $\beta=8$ or $\beta=-2$, and the spheres have the following equations:

$$\left(x-\frac{8}{5}\right)^2+\left(y+\frac{8}{5}\right)^2+\left(z-\frac{16}{5}\right)^2=\frac{289}{25} \text{ and } (x+2)^2+(y-2)^2+(z+4)^2=49.$$

Conclusion

We can see that the pencils of planes or of spheres may be useful for solving some problems in analytic geometry in 3-dimensional Euclidean space. In addition, there exist dynamic geometry environments like GeoGebra to visualize both planes and spheres, and also the linear combination of their equations. Although the examples given in this paper are related to the planes and spheres, the method of pencils is also applicable to other surfaces in space.

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Using a Non-Traditional Activating Method in Mathematics Education

Kristína Cafiková[a*] – Anna Hrešková[a] – Edita Szabová[a]

[a] *Department of mathematics, Faculty of Natural Sciences, Constantine the Philosopher University in Nitra, Tr. A. Hlinku 1, SK - 949 74 Nitra*

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Abstract

The contribution deals with the description and practical use of one non-traditional activating method from Germany. The name of this method is Silent mail. During the research in which we implemented this method in mathematics education to six-graders, we performed an indirect observation of a part of the class, with parental consent. This indirect observation was evaluated with the use of statistical implicative analysis. The contribution includes several samples of pupils´ works and their content analysis. The overall research implies that the activating method improves argumentative skills and constructivist thinking of pupils.

Keywords: Activating method, teaching mathematics, statistical implicative analysis.

Classification: D40

Introduction

The roots of mathematics education using teaching methods date back to the ancient Greece, when mathematicians of this period drew in the sand using sticks and illustrated their calculations using pebbles. At present, these methods would be obsolete and ineffective. Nowadays, there are different teaching methods, approaches and activities designed for an easy adoption, understanding and fixation of the curriculum by the pupils. One of the many countries that implements a great number of teaching methods used not only in mathematics is Germany. Individual teaching methods have been built in the teaching process in primary and secondary schools in Germany for several years. One of these methods is also Silent mail.

Activating Method Silent Mail

The name of the method comes from the German "Stille Post". Pupils are divided into groups of five to seven. An elected pupil gets a paper with an impulse. He is the only one who can see it. The impulse can be in a form of a written text, thought, formula, word or a drawing. The pupil has to write or draw the first thing that comes into his mind after seeing the impulse. When finished, he folds the paper in a way that only his solution is visible and sends it to another pupil. This one will look at the solution of the previous one and this solution will become a new impulse for the second pupil, who has to put down his reaction, then fold the paper and pass it to another pupil. This process repeats until each pupil in the

*Corresponding author: kristina.cafikova@ukf.sk
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group writes or draws something. After the round is finished, they unfold the paper. In this way, pupils can see the initial impulse, the whole process and the final solution they reached. In case that each group got the same initial impulse, the groups can compare their solutions. The use of Silent post helps pupils to understand also the information they have learned a long time ago. The feeling of success can't subsequently lead also to the interest of pupils in mathematics, as primary school pupils often associate the teaching method and the subject being taught, mathematics. This interest will then motivate them to deal with mathematics also in their free time (www.studiensemina-koblenz.de; Barzel, Büchter, Leuder, 2011).

The Activating Method Used in Practice

The research was realized in school term 2013/2014 in class 6.B at primary school in Benková street in Nitra. There are 23 pupils I the class, but we obtained parental consent to video- record only from 10 pupils.

At the beginning of the lesson we explained the correct use of the teaching method and explained how the paper will move between pupils. The pupils in the last row got papers with the pre-prepared impulses. At first they were unsure what to do, but they quickly understood. As there were only three papers in the class with 23 pupils, the ones who were not writing began to disturb the class and talk. Therefore, we spontaneously decided that we would not wait until each paper gets to each pupil, but only waited until each paper reached the last pupil in the row and then began to analyze their work (Figure 1 to 3).

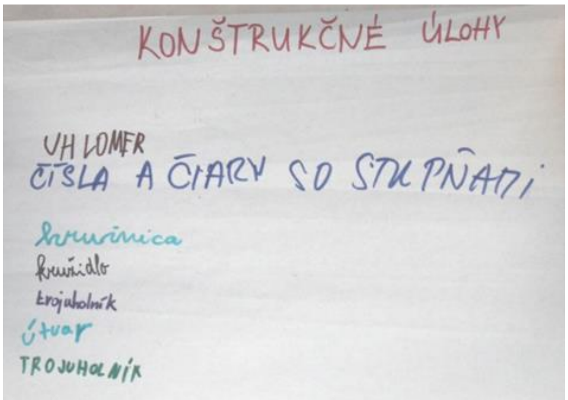
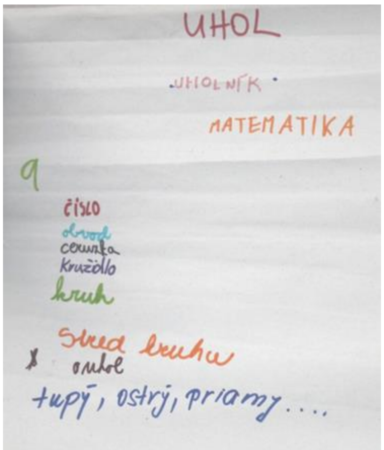


Figure 1: *Implementation of Silent post 1*



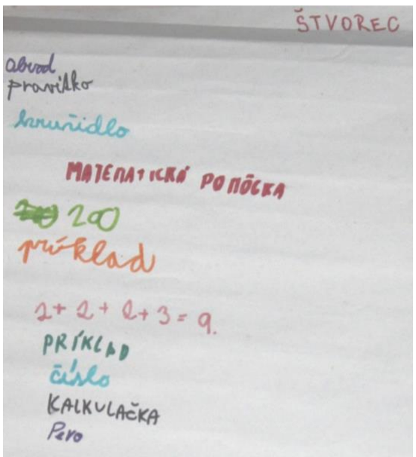


Figure 2: *Implementation of Silent post2* ***Figure 3:*** *Implementation of Silent post3*

Also pupils who were not paying attention calmed down and started listening. After looking for associations between the first two terms also pupils entered the analysis. Individual pupils raised their hands and explained the associations between the lines.

The experiment was followed by the analysis of the video and we decided to use the statistical implication analysis developed by *Régis Gras* (2008) to evaluate the results. We selected the following didactic variables in terms of application of non-traditional methods in mathematics education:

T1 Pupil was solving the task;

T2 Pupil proceeded in solving the task according to the teacher's instruction;

T3 During solving the problem, pupil abided by the rules put out by the teacher;

T4 Pupil cooperated on the solution with classmates in the same group;

T5 Pupil discussed the solution with others in the group in a written form;

T6 Pupil required checking of a partial solution;

T7 Pupil was disciplined during the lesson;

T8 Pupil expressed dissatisfaction with the course and form of the activity;

T9 Pupil was active during the lesson;

T10 Pupil refused to participate in the task solving;

T11 Pupil dispassionately observed the happening around;

T12 Pupil engaged in the whole-class discussion about solutions.

The application of the Silent post led to the following implication rules:

$\big(\mathrm{t6} \rightarrow (\mathrm{t2} \leftrightarrow \mathrm{t12})\big)$ cohesion of 0,808;

$\big(\mathrm{t8} \rightarrow (\mathrm{t10} \rightarrow \mathrm{t5})\big)$ cohesion of 0,507;

$(\mathrm{t9} \rightarrow \mathrm{t3})$ cohesion of 0,355.

In the first implication tree (Fig.4) we can see that a part of the pupils abided by the rules put out by the teacher, discussed the solution in an oral form with the whole class, but needed a check of accuracy of their reaction. The second implication tree represents a group of pupils who discussed the solution with their classmates in a written form, but declined to participate in the task solving, while they expressed dissatisfaction with the course and form of the activity. The third implication tree represents pupils who were active and respected the teacher's rules.

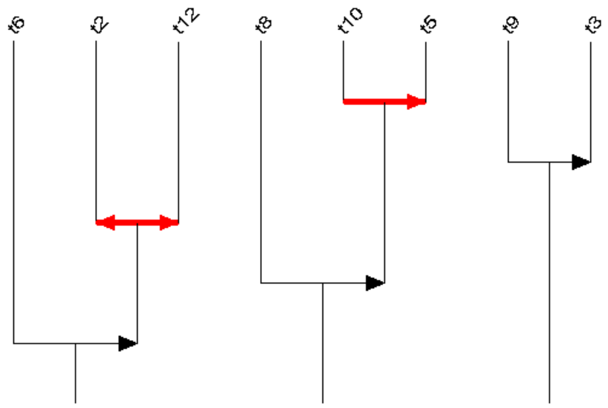


Figure 4: *Implication tree*

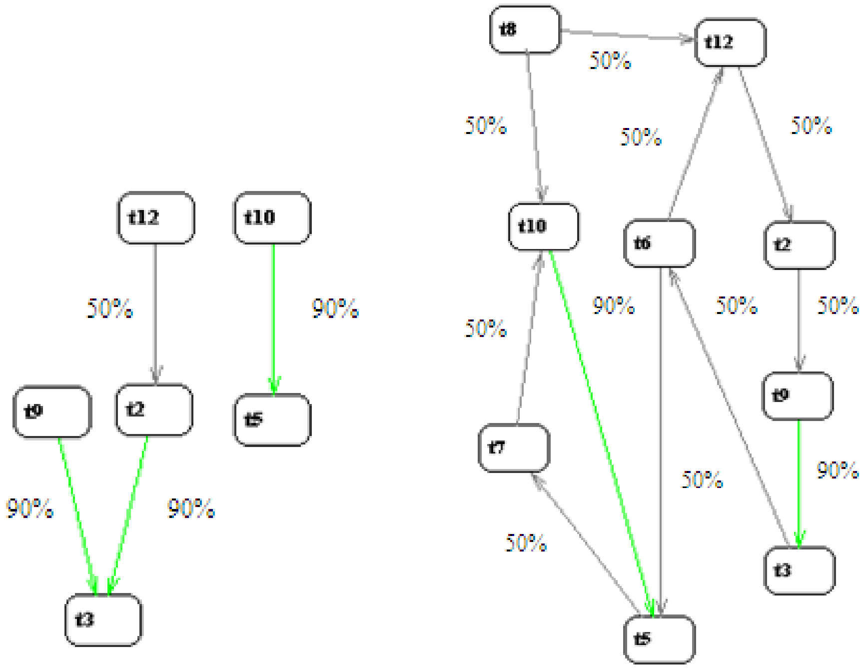


Figure 5: *a) Implication graph with high set values; b) Implication graph with low set values*

The implication graph (Fig.5) is more complicated. We set high limits and created two graphs: a graph with vertices T12 (Pupil engaged in the whole-class discussion about solutions), T9 (Pupil was active during the lesson), T3 (During solving the problem, pupil abided by the rules put out by the teacher), T2 (Pupil proceeded in solving the task according to the teacher's instruction) and a graph with vertices T10 (Pupil refused to participate in the task solving), T5 (Pupil discussed the solution with others in the group in a written form). After lowering of the limit values we obtained one comprehensive graph with nine vertices, where the vertex T9 (Pupil was active during the lesson) popped out between vertices T2 and T3. There also originated an edge between vertices representing didactic variables T2 and T9. We can notice the association between the didactic variable T8 (Pupil expressed dissatisfaction with the course and form of the activity), T12, T10 and T5. One group of pupils expressed their dissatisfaction with the course and form of the activity. This group subsequently divided into two. One part of the dissatisfied pupils declined the participation in task solving, but discussed the solution with their classmates in a written form. The second part of the dissatisfied pupils engaged in the whole-class discussion. This implies that though some pupils expressed dissatisfaction and maybe even refused to participate in the task solving, they still discussed the solution either with classmates or with the whole class.

Conclusion

Teaching methods are an active element of the teaching process; therefore it is important how we understand the term teaching methods. According to *H. Meyer*, "teaching methods are forms and approaches in which and through which teacher and pupils acquire the surrounding natural and social reality in institutional conditions" (Meyer, 2005, p.45).

The activating method Silent mail is an unconventional method that supports constructivist thinking of pupils and their argumentative skills. Therefore we think that its implementation in the teaching process of mathematics in primary and secondary schools could positively influence the mathematics education in Slovakia.

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Analysis of Errors in Student Solutions of Context-Based Mathematical Tasks

Antal Csáky[a*] – Edita Szabová[a] – Zuzana Naštická[a]

[a] *Department of mathematics, Faculty of Natural Sciences, Constantine the Philosopher University in Nitra, Tr. A. Hlinku 1, SK - 949 74 Nitra*

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Abstract

The submitted contribution is concerned with analysis of errors made by students when solving context-based mathematical tasks. The contribution comprises evaluation of four tasks which were designed by the authors of the contribution within project KEGA 015 UKF – 4/2012 in Slovakia. Altogether 56 first and second year students of primary teacher training university master programme were asked to solve the tasks. The errors in the student solutions were identified and classified primarily following Newman´s error categories and additional categories suggested by the authors of the contribution, who furthermore propose 13 error subtypes. In total 127 inappropriate solutions of the four tasks were included in the evaluation. The authors present a sign scheme and a correspondence map of student errors based on statistical analysis. As evidenced by the analysis, students make similar errors when solving tasks of the same type. The objective of the authors is to identify accurately and classify the error types occurring in student solutions.

Keywords: Context-based mathematical tasks, student errors, KEGA 3700109.

Classification: 97D70

Introduction

The importance of relating mathematics education to everyday life is commonly known and generally accepted. In labour market employers are often disappointed by the graduates' inability to use mathematical knowledge in practice, and thus they demand that more emphasis be put on practice-oriented mathematics education (Graumann, 2011). The main objective of this movement is to develop students' ability to apply mathematics in everyday life, which is seen as the core goal of mathematics education (Biembengut, 2007).

The use of mathematics in everyday life has also been among the main concerns of researchers within project KEGA 015 UKF – 4/2012 (Fulier et al., 2014) at Constantine the Philosopher University in Nitra, Slovakia. The research project, as well as the Programme for International Student Assessment (PISA) organized by the Organisation for Economic Co-operation and Development (OECD), has been interested in applications of mathematics in daily life. In PISA tasks there are used such real word problems which require quantitative reasoning, spatial reasoning or problem solving (OECD, 2003). The Slovakian KEGA project researchers also based their priorities on the above mentioned ideas. Therefore the main focus of the project was on designing appropriate learning material – new context-based tasks for pupils at elementary schools covering primary and lower secondary education

*Corresponding author: antal.csaky@ukf.sk
DOI: 10.17846/AMN.2015.1.1.68-75

levels, inspired by PISA tasks. Despite the importance of contexts for learning mathematics, several studies (Wijaya, Heuvel-Panhuizen, Doorman and Robitzsch, 2014) indicate that contexts can also be problematic for students when they are used in mathematics tasks. In addition, if students are expected to be able to use mathematics in practical life situations, teachers have to be well-trained in this area of mathematics teaching.

Within KEGA project the research was conducted among hundreds of elementary school pupils in Slovakia, and the concern of measurements was their ability to solve context-based mathematical tasks. Although the results were evaluated through common statistical methods (Csáky – Cafiková, 2015), no similar analysis of errors was performed. We believe that the presented analysis can provide us with additional useful information about the difficulties that pupils and students face when solving mathematical tasks of this kind.

The concern of the submitted contribution is to identify the errors made by teacher trainees in their solutions of contextual mathematical tasks.

Theoretical background

There are many ways to define context in mathematics education. Generally, it can be stated that „contexts are recognized as important levers for mathematics learning because they offer various opportunities for students to learn mathematics“ (Wijaya et al., 2014). PISA study defines context as a specific setting within a ´situation´ which includes all detailed elements used to formulate the problem (OECD, 2003). The ´situation´ refers to the part of the students´ world in which the tasks are placed. The term context-based mathematics task refers to a task situated in real-world setting which provides elements or information that need to be organized and modelled mathematically (Wijaya et al., 2014).

Analyzing student errors in solving context-based mathematical tasks

The analysis of student errors in solving context-based mathematical tasks can be done with the use of Newman's model, the Newman Error Analysis (1977, 1983). This model uses five categories of errors based on the process of solving mathematical word problems: errors of reading, comprehension, transformation, process skills, and encoding. In the submitted contribution this model is compared with other models, such as stages of Blum and Leiss´ modelling process (Maass, 2010) and the PISA mathematization stages (OECD, 2003), which were designed for solving context-based mathematics tasks. After detailed comparison, Wijaya et al. (2014) considers Newman's model to be suitable for analysis of student errors in solving context-based mathematical tasks (Wijaya et al., 2014).

Research question

Within the Slovakian KEGA project several hundreds of context-based mathematical tasks were designed for elementary school pupils, and the tasks were also tried and tested in elementary school environment. The project results indicate that the designed tasks are appropriate for the purpose of improving mathematical literacy of elementary school pupils.

The submitted contribution aims to investigate results of teacher trainees for primary education level who took a test designed for pupils of the last grade in lower secondary education level, identify and evaluate the errors the teacher trainees made when solving the context-based tasks. The presented statistical approach can be further used for the final evaluation of test results obtained within KEGA project.

The research question is as follows:

What types of errors do teacher trainees for primary education level make when solving context-based mathematics tasks?

Method

Mathematical tasks

The test used for the research was administered to collect data about student errors when solving context-based mathematical tasks. The test was designed on the basis of PISA tasks and KEGA project tasks. The test consisted of five tasks. The tasks covered several mathematical issues. In order to solve the tasks common mathematical knowledge from elementary school curriculum was sufficient. Although the test consisted of five tasks, four of them were covered in the research evaluation. The one task, focused on elementary statistics and probability, was not covered in the evaluation since it was impossible to identify accurately types of student errors. The first tasks, titled *Smartphone* required elementary knowledge of geometry and measure. The second task, titled *SMS*, covered elementary curriculum about numbers, variables and computation. The third tasks, titled *Electricity*, tested students' elementary knowledge of relations, functions, tables and diagrams. The fourth evaluated task, titled *Cars*, was focused on elementary logic, reasoning and proofs in mathematics. Students were allowed to solve the tasks for 35 minutes in total, similarly to KEGA project within which elementary school pupils had from 7 to 8 minutes for a task.

Participants

Altogether 56 first and second years students of primary teacher training university master programme took the test. After finishing their studies, the students will teach also mathematics in primary education level. During the spring term in 2015 they attended an optional subject at Department of Mathematics, Faculty of Natural Sciences, Constantine the Philosopher University in Nitra, within which they were ask to solve the context-based mathematical tasks. The students were not informed about the test in advance, since the objective was to examine their ability to use their mathematical knowledge when solving context-based tasks without any special preparation.

Procedure of coding the errors

To investigate the errors, only the students´ incorrect responses, i.e., the responses with no credit or partial credit, were coded. Missing responses, which were also categorized as no credit, were not coded and were excluded from the analysis because a student error cannot be identified from a blank response. For the analysis of student errors the model proposed by Wijaya et al. (2014) was used, based on Newman´s error categories and in agreement with Blum and Leiss´ modelling process and PISA´s mathematization stages (see Table 1).

Table 1

Error type	Sub-type	Explanation
Comprehension	A – Misunderstanding a keyword	Student misunderstood a keyword, which was usually a mathematical term.
	B – Error in selecting information	Student was unable to distinguish between relevant and irrelevant information.
Transformation	C – Procedural tendency	Student tended to use directly a mathematical procedure without analyzing whether or not it was needed.
	D – Taking too much account of the context	Student´s answer only referred to the context/real world situation without taking the perspective of the mathematics.
	E – Wrong mathematical operation/concept	Students used mathematical procedure/concepts which are not relevant to the tasks.
	F – Treating a graph as a picture	Student interprets and focused on the shape of the graph, instead of on the properties of the graph.
Mathematical Processing	G – Algebraic error	Error in solving algebraic expression or function.
	H – Arithmetical error	Error in calculation.
	I – Error in mathematical interpretation of graph	Student mistakenly focused on a single point rather than on an interval.
	J – Improper use of scale	Student could not select and use the scale of a map properly.
	K –Measurement error	Student could not convert between standard units or from non-standard units to standard.
	L – Unfinished answer	Student used a correct formula or procedure, but they did not finish it.
Encoding	M	Student was unable to correctly interpret and validate the mathematical solution in terms of the real world problem.

Statistical analyses

The objective of the analyses was to identify types of errors in relation to specific mathematical tasks. The aim was to find out what error types are typical for which tasks in the test. Data were evaluated with the use of $\chi^2 -$ test of independence, correspondence analysis and sign scheme.

Two nominal variables were used in the statistical analyses – the first variable assuming m categories (m error types) and the second variable assuming n categories (n evaluated tasks). Each task was assigned a score (number of students who made an error of specific type in the task), so that the contingency table $m \times n$ would be formed.

In total, we obtained 224 responses (number of tasks solved by all students in total), which included 82 correct responses (36%), 127 incorrect responses (i.e. no credit or partial credit, 57%), and 15 skipped tasks (7%). The error analysis was carried out for the 127 incorrect responses. The analysis revealed that 57% of the 127 errors were mathematical processing errors, and 25% were transformation errors. Encoding errors (11%) and comprehension errors (7%) were less frequent.

For a detailed list of identified errors according to error sub-types see Table 2.

Table 2

Sub-types	Smartphone	SMS	Electricity	Cars	Active Margin
A	1	3	2	0	6
B	1	2	0	0	3
C	6	0	0	1	7
D	0	3	0	0	3
E	0	6	0	1	7
F	15	0	0	0	15
G	13	2	0	0	15
H	2	0	0	2	4
I	1	0	27	1	29
J	0	0	17	0	17
K	0	0	0	0	0
L	3	4	0	0	7
M	0	0	1	13	14
Active Margin	42	20	47	18	127

The standard tool for testing dependence in contingency table is $\chi^2 -$ test. However, its result does not reveal any specific information about the structure of the variables. Correspondence analysis (CA) is a descriptive and survey method, it does not cover any tools for testing the statistical significance of the obtained models. In quantitative research, CA can be used as a part of any stage of the categorical variables processing – from preparation to the presentation of results. (Hebák et al., 2007, p. 169). CA serves to reduce multidimensional space of line and column profiles (see Řezanková, 2007) in the table to a space of two dimensions, if possible, so that relations between categories of the quantities classified in the contingency table can be shown as legibly as possible – in a planar graph. The closer the points are to each other in the correspondence map, the higher the similarity between the corresponding categories. Groups of similar categories can be also identified regarding the position of the points with respect to the main axes (Spálová, Fichnová, Szabová, 2013).

Sign scheme is another way of graphical representation showing the relations in the contingency table. It is based on testing the fit between the observed and the expected frequencies. The sign scheme is formed on basis of the normalized residuals (Hendl, 2006). The greater their absolute values, the more intense the relation between the categories of the variables. Normalized residuals are transformed to mathematical signs (Řehák – Řeháková, 1978). Signs $+$ mean that the pair of the two categories (the error type and the task) evaluated in the cell of the contingency table is over-represented, signs $-$ mean that the pair of the two categories evaluated in the cell is under-represented (Szabo, 2015).

Results

The tested null hypothesis H_0 was that the type of an error does not differ with respect to the type of the solved task. The significance level of the hypothesis testing was $\alpha = 0{,}05$. The statistical testing was processed with the use of SPSS software. The results of the statistical processing were as follows: the value of the test statistic $\chi^2 = 272{,}75; p = 0{,}000$. The tested null hypothesis is thus rejected at significance level $\alpha < 0{,}01$. It means that the type of an error depends on the type of the task. Accordingly it makes sense to do a correspondence analysis for the data. The first dimension contributes to the overall inertion

(2,148) by 40,7 % of the total inertion, the second dimension by 32,7 % of overall inertion that together sum to 73,4 % of the overall inertion, so they together contain sufficient information about the correspondence of row and column categories. The correspondence map (Figure 1) and the sign scheme (Table 2) serve to illustrate this dependence.

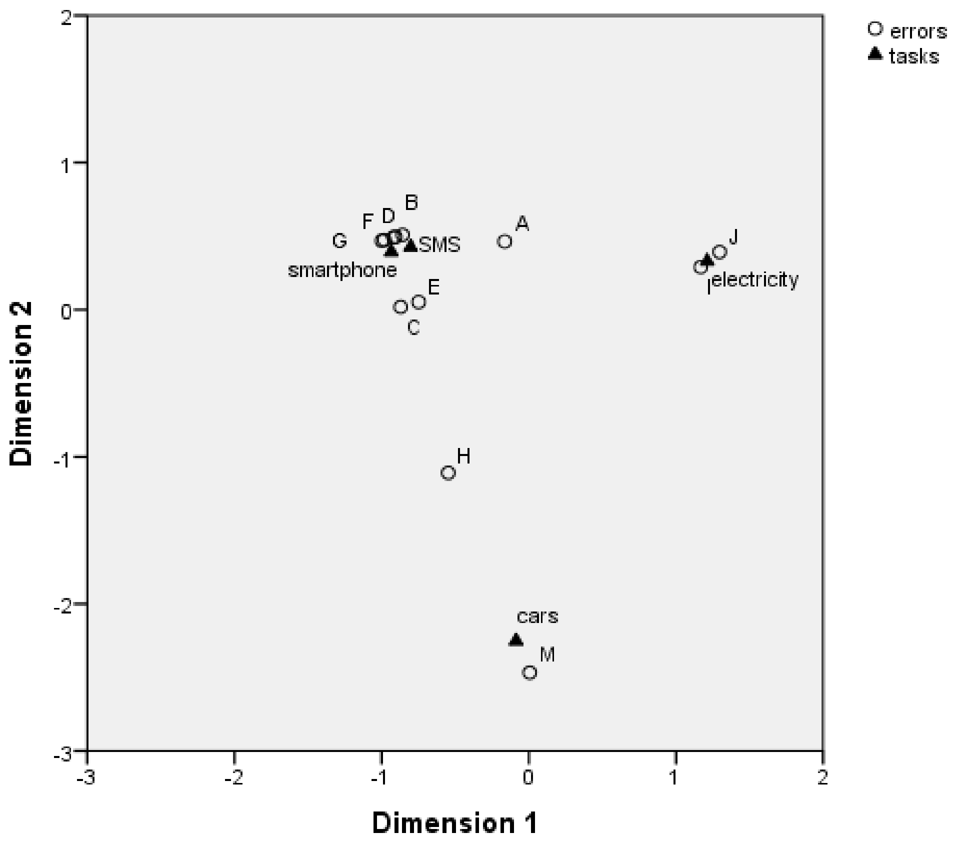

Figure 1: *Correspondence map*

Table 2: *Sign scheme*

	Smartphone	SMS	Electricity	Cars
A	0	+	0	0
B	0	+	0	0
C	++	0	-	0
D	0	+++	0	0
E	0	+++	-	0
F	+++	0	--	0
G	+++	0	--	0
H	0	0	0	+
I	---	--	+++	0
J	--	0	+++	0
L	0	++	-	0
M	--	0	-	+++

The statistical results show that when solving the first task, which was focused on curriculum covering geometry and measure, students significantly often made errors of F, G and C type. Students drew unsuitable pictures, used inappropriate method for computation, or did not use Pythagorean Theorem. The second most frequent error made when solving this task was using an inappropriate formula without any drawings and without any mathematical considerations, which led to an incorrect result.

In the task focused on curriculum about numbers, variables and computation, students were asked to determine the most advantageous monthly fee programme offered by the phone call service provider. The most frequent errors were of comprehension and transformation type: A, B, a D, E. For students it seemed difficult to identify crucial information and set appropriate mathematical conception in order to determine the most favourable monthly fee programme. Many students answered the question without any mathematical considerations. The second most frequent error (type L) was leaving the task unfinished after having performed correct computation, in other words students forgot to answer the question asked in the task wording.

In the third task, which was focused on relations, functions, tables and diagrams, students made the most errors, and the errors were of type I and J. Students were expected to gain information from the graph illustrating electricity consumption. Students were not able to comprehend the scaling in the graph, and thus failed to determine the intervals which would provide them with the answer to the question.

In the fourth task, which was focused on elementary logic, reasoning and proofs in mathematics, the least errors were recorded. The recorded errors were of type M. Students were not able to interpret their results. Students were expected to work with data about car theft rate in Slovakia during recent years. Although many students managed to compare the numbers correctly, they failed to interpret their results. Some students made minor arithmetic mistakes when trying to express the data in percents (error type H).

Conclusion and Discussion

In the presented research the analyses focused on students´ errors when solving context-based mathematics tasks. The objective was to determine if primary teacher trainees are able to use mathematics in daily life situations. The analyses show that when solving tasks of specific type students make similar errors. The types of errors occurring in specific task types have been determined and categorized. As the error analysis shows, the most errors were made by students when solving tasks examining their ability to interpret graphs and diagrams and in geometrical tasks, namely the errors were of type F, G, and I, J, which – apart from error G (algebraic error) – can be reduced by solving context-based mathematics tasks within school instruction more frequently. The authors of the submitted contribution believe this to be of utmost importance, since the ability to gain information from graphs and diagrams and to become conscious of crucial details is getting more and more needed in real life situations.

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Project "E-matik+" Education for Mathematics Teachers

Andrej Ferko[a] – Lilla Koreňová[a*] – Štefan Solčan[a]

[a] *Department of Algebra, Geometry and Didactics of Mathematics, Faculty of Mathematics, Physics and Informatics, Comenius University in Bratislava, Mlynská dolina, SK - 842 48 Bratislava*

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Abstract

From the point of view of effective education, improving the competences of mathematics teachers is a necessary process. The form of their education is just as important as the content of their further education. On the Faculty of Mathematics, Physics and Informatics, Comenius University, we are providing accredited continual education for teachers of mathematics in e-learning form for several years, from the area of geometry and didactics of mathematics. Through this distance method, their digital competencies are being improved, as well. This contribution is describing the continual education from its content and formal perspective, as well as experience regarding educating teachers.

Keywords: Teacher education, digital technology, e-learning.

Classification: D40, U50, U70

Introduction

Digital technologies (DT) offer teachers a possibility to make use of new educational methods, e.g. the constructivist approach, controlled search, workshop method or peer instruction method. DT are very suitable for project teaching, too. Teachers can make use of blended learning, flipped classroom method, etc.

Last but not least, the computers are used for electronic testing when knowledge of the pupils is measured. Generally accepted theoretical basis in this area is Technological Pedagogical Content Knowledge (TPACK) as a framework for the integration of technology within teaching.

According to (Mishra, P. & Koehler, M. J., 2006), see Figure 1, teachers must understand how technology, pedagogy, and content interrelate, and create a form of knowledge that goes beyond the three separate knowledge bases. Teaching with technology requires a flexible framework that explains how rapidly-changing, protean technologies may be effectively integrated with a range of pedagogical approaches and content areas (Mishra, P. & Koehler, M. J., 2008).

*Corresponding author: korenova@fmph.uniba.sk
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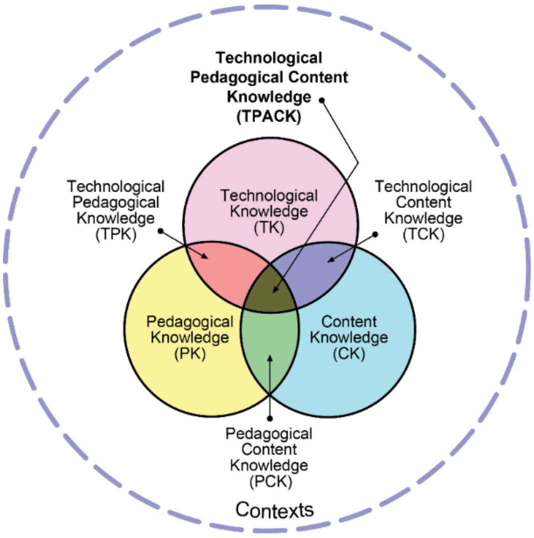

Figure 1 *(Technological Pedagogical Content Knowledge: A new framework for teacher knowledge, 2006)*

New technologies are driving necessary and inevitable change throughout the educational landscape. Effective technology use, however, is difficult, because technology introduces a new set of variables to the already complicated task of lesson planning and teaching. The TPACK framework describes how effective teaching with technology is possible by pointing out the free and open interplay between technology, pedagogy, and content. Applying TPACK to the task of teaching with technology requires a context-bound understanding of technology, where technologies may be chosen and repurposed to fit the very specific pedagogical and content-related needs of diverse educational contexts (Mishra, P., Koehler, M. J., & Kereluik, K., 2009).

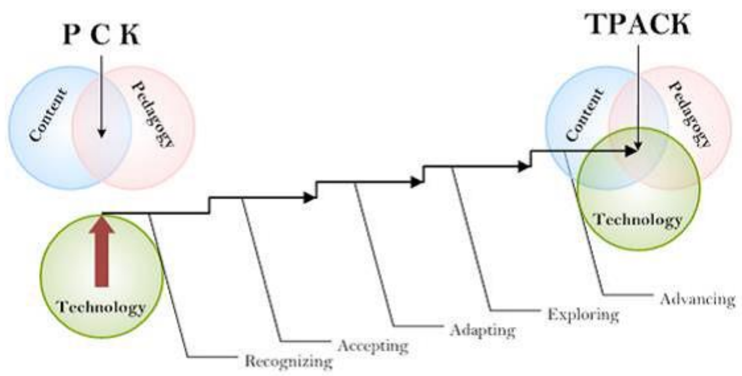

Figure 2 Mathematics teacher TPACK standards and development model. (Image from http://www.citejournal.org/vol9/iss1/mathematics/article1.cfm)

The questions of TPACK and mathematics teaching were in detail dealt with by Niess et al. Figure 2 shows the visual description of teacher levels as their thinking and understanding merge toward the interconnected and integrated manner identified by TPACK.

Besides benefits of DT for teaching, the computers are used for electronic testing when knowledge of the pupils is measured.

During mathematics classes, pupils can make use of digital technology in various ways:

- during numerical calculations so they can concentrate on the solution of the problem itself;
- for visualisation, modelling and simulation of problems and thus to obtain such a graphical representation of the problem, which pushes them towards a solution;
- as a source of educational materials e.g. e-books or videos, interactive educational materials;
- drilling exercises, a pupil can make use of electronic working sheets or e-tests to evaluate himself.

From a didactic point of view, we can use these resources in different teaching methods:

- method of controlled discovering,
- project education (this is another rising form of mathematics teaching, which is gaining new possibilities in the digital environment. Students use mostly cooperative learning, which is enabled by m-learning),
- peer instruction method (in this case, the tablet is used as a voting device),
- "The flipped classroom" method, etc.

From an educational point of view, teaching with a tablet is appropriate for both individual and collective work.

According to (Lukáč, S., 2001) the use of a computer while teaching can contribute to fulfilling some didactic functions:

- motivation - the computer can contribute to forming positive attitude towards learning
- informative function
- driving function - highly interactive educational programs can control the exchange of opinions between the subject of education and the computer
- rationalization function - proper integration of the computer into the educational process can support the differentiation of procedures and methods of teaching regarding the relations towards individual students
- control function - using the computer can make diagnosis and evaluation of educational results much more effective
- communication function - computer aided teaching supports mutual communication between students
- social function – work in groups.

At the present, it is important for teachers of mathematics in Slovak schools, that they improve their digital competencies, expand their knowledge from selected domains in mathematics and get acquainted with new educational methods and forms. From the digital technologies point of view, the following tools are suitable for teaching mathematics: GeoGebra, HotPotatoes, software for interactive whiteboards, e.g. OpenSankore, which is open source, etc. Familiarization with these tools is the content of some modules of continual education. For the distance form of education we chose LMS Moodle. Even the communication via LMS Moodle is improving their digital competencies.

Project “EMATIK+”

At Comenius University, we have dedicated already 10 years for preparing future teachers of mathematics and the topic of using ICT in teaching mathematics on elementary and secondary schools. For this purpose, two courses are available. In the first course, students are acquainted with the wide range of software (and other ICT means), which are available on elementary and secondary schools in Slovakia. In the second course, they are acquainted with the didactics of teaching mathematics using ICT. There are several dozens of diploma works with this topic. Through continuing education of teachers, during the years 2006 – 2009, with the help of the ESF EU „EMATIK” grant, we trained more than 1000 teachers in the area of using ICT in education via e-learning. Based on a good experience with e-learning form of education, in 2012 we accredited continuing education as a course “Digital technologies in teaching mathematics on elementary and secondary schools”. At the present, within the project KEGA, we are testing new forms and methods of this continuing education. The success of the EU depends on the skills of workers and the creative economy for development of information society. Continual vocational training has by CEDEFOP European Centre double meaning: it supports economic growth and social cohesion. The project "E-matik +, lifelong training of teachers of mathematics" (KEGA 094UK-4/2013) solved by two departments at Comenius University, is to create a Moodle e-learning portal, including selected video-tutorials, adapted according to feedback from current practice in three recently accredited lifelong education programs for teachers of mathematics in Slovakia:

1. Digital technologies in mathematics at primary and secondary schools
2. Geometry and computer graphics in the further education of teachers of mathematics and its use in primary and secondary schools (GaPG)
3. Creating teaching and testing in digital form

Short title of the project is created after keywords Mathematics, ICT, e-learning. The plus sign represents the main added value – improving professional education. We plan to optimize the cost of software tools (open information technology, global movement FOSS), travel costs of emerging social network of teachers, following the successful ESF project EMATIK (finished 2008). Our long-term goal is to prepare the international cooperation and follow-up by for English or German language areas. Educational materials will be created, verified and tested in practice with a unique target group – teachers of mathematics who directly co-create the quality of education our children and youth.

The necessity to improve professional competences of mathematics teachers (both in digital technologies and new didactic approaches) is a result of practice needs as well as the goals in the new state education program. We plan to create a digital content highly adapted to the recent progress, easy to access and very modern.

The goals of continuing education are focused on perfecting the professional competences in the field of digital technologies, necessary for standard pedagogic work (within teaching mathematics on elementary and secondary schools). The goal is to develop and innovate the competences of teachers regarding digital technologies. The educational program is a summary of independent modules. GeoGebra plays a key role in two of them:

Module 1: Freely available software in teaching mathematics and methodics of its use in teaching mathematics on secondary schools (GeoGebra, HotPotatoes, Open Sankore, etc.)

Teachers learn how to properly embed software into the educational process, to assess the suitability of the form of the used software and they gain an overview about the accessible software and its placement into education

Module 3: Dynamic mathematics and methodology of its use in teaching mathematics on secondary schools (GeoGebra, Cabri geometry, etc.)

Teachers learn how to use the GeoGebra software and Cabri geometry in teaching mathematics on secondary schools. They learn how to properly embed digital technologies into the educational process and they will be able to apply the software according to the latest forms of teaching mathematics and to effectively use the given software in a digital classroom.

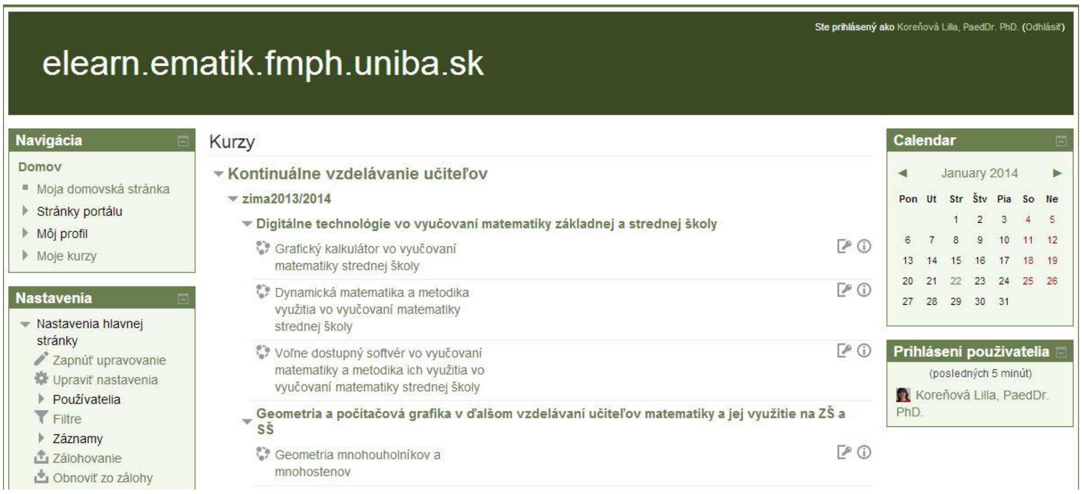

Figure 3 *The continuing education portal, available at http://elearn.ematik.fmph.uniba.sk.*

The project "E-matik +" aims to create and use Moodle-based e-learning portal. The education itself is planned for one semester and it is concluded with final presentation and a discussion on the topic of the completed course in front of a three-member examination commission.

We present a few examples of tasks, presented in the courses.

Example 1

A significant part of the module „Projective geometry“ of the education program GaPG is dedicated to finite structures – affine and projective planes. We try to explain some basic notions and properties at the pupils of primary and secondary schools level. Our goal is to find some connection(s) between affine and projective planes and the real life. As an example of useful application is as follows:

In a firma a computer net with local subnets is needed with some requirements (for optimalizing and security):

- every two computers must be connected using exactly one local subnet,
- for every two subnets there is exactly one computer being connected in both subnets,
- there exist at least four computers of which no three are in the same local subnet.

How much computers must our firma buy in order to build this net if they need at least 15 and can afford less than 35 computers? Without knowing that it is a model of finite

projective plane one can just guess and increasing the number of computers makes this problem more difficult.
The similar, more general and more discussed is the „communication problem", where a set of users would like to communicate with each other (e.g. is participants of a telephone system). After having formulated the requirements for a communication system, we translate this into geometric language. This leads to notions of „linear space" and – as a special case – the „finite projective plane" which correspond (depending on the requirements) to those communication systems. (See e.g.: Albrecht Beutelspacher And Ute Rosenbaum: Projective Geometry: From Foundations to Applications. Cambridge University Press 1998, ISBN 052148277 1 hardback, ISBN 0 521 48364 6 paperback, available at http://www.maths.ed.ac.uk/~aar/papers/beutel.pdf).
Secondary school teacher Kinga Horvath – participant of the above mentioned e-learning course – had a final presentation on finite affine planes with this example (Horvath, K., 2001):
There were 16 racers registered to a motorcycle race in which only 4 can use the competition path. Organize groups of 4 racers in such way that every two racers will fight in competition exactly one times.
In geometric language – if the racers are „points" of finite (affine) plane then the starts (groups) of the competition are „lines" of this plane.
After discussing how to solve this real-life problem one can get this table of starts:

1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.	16.	17.	18.	19.	20.
0	4	8	12	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
1	5	9	13	4	5	6	7	5	4	7	6	6	7	4	5	7	6	5	4
2	6	10	14	8	9	10	11	10	11	8	9	11	10	9	8	9	8	11	10
3	7	11	15	12	13	14	15	15	14	13	12	13	12	15	14	14	15	12	13

This can be (as a model of a finite affine plane) illustrated by a picture (Figure 4)

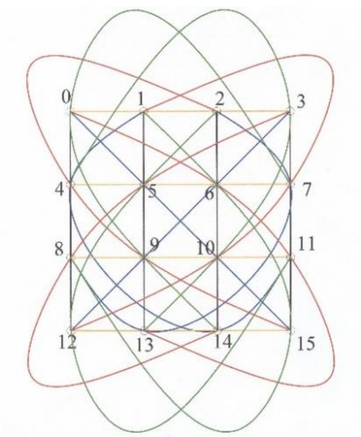

Figure 4 *Modelling the solution by Kinga Horvath*

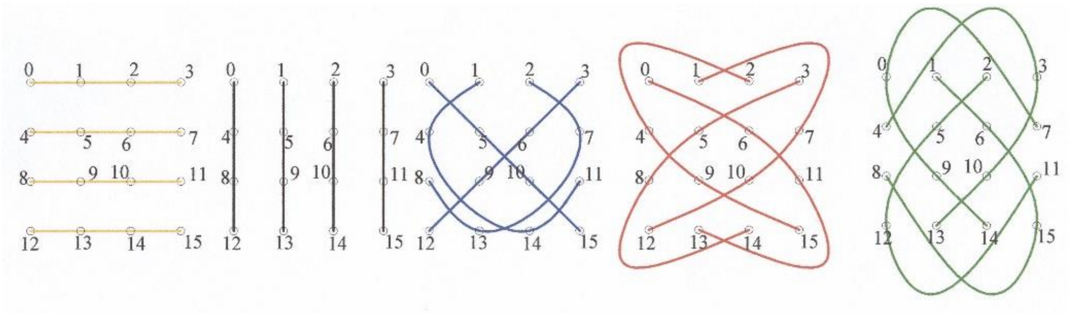

Figure 5 *Modelling the solution by Kinga Horvath, part 2.*

Example 2

During the course Digital Technologies in Mathematics at Primary and Secondary Schools, teachers get acquainted with useful Java applets, which they can use during the explanation of educational materials on an interactive whiteboards or as an individual task for pupils working with a computer (see Figure 6).

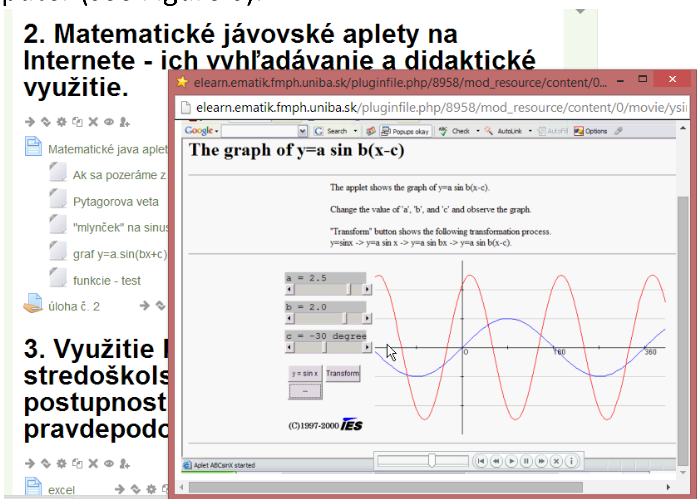

Figure 6 *A screenshot of an interactive Java applet activated.*

In this course, teachers of mathematics also get acquainted with the open source software GeoGebra, which has a very wide range of use in school mathematics. They get an overview on how to create GeoGebra files and applets. In addition, the learn how to use this software on concrete problems in different didactic situations and in different educational methods and forms. We are presenting an example:

We would like a big yard for our pet!

Siblings Eva and John would like to build a fence for their pet's yard. Their parents allowed them to build a fence around an area in the shape of a rectangle next to the stable, but they have only 15 meters of netting. What should be the size of the paled area if they want to have it as large as possible for their pet to play on? See picture. If we choose x as one side of the rectangle, then we get the function of the area y depending on the side x.

$$y = x \cdot (15 - 2x)$$

How will the values of x and y look if we can measure only with a precision of 1m? 0.5m? 0.1m? Let's make a table. Where will be the maximum for the area?

The solution using GeoGebra is presented in Figure 7. This example can be used as a motivational task on an interactive whiteboard, during which the teacher uses the controlled discussion method and a solved GeoGebra worksheet. A different use of this very example is, if the teacher uses the guided discovery method and students work in pairs next to a computer. Here students solve the problem by themselves in GeoGebra and discover some soundness – the function, its graph and maximum.

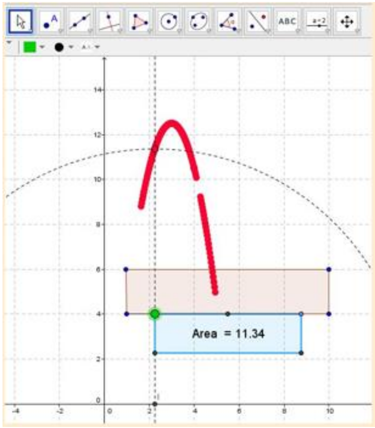

Figure 7 *The size of area maximum found!*

A popular software for teachers is also the free software HotPotatoes. Using this software, teachers can create electronic tests. E-tests are not only considered as a tool for measuring the knowledge of the students, but mainly as electronic worksheets, which enable the students to advance individually while solving problems and with immediate feedback (Figure 8).

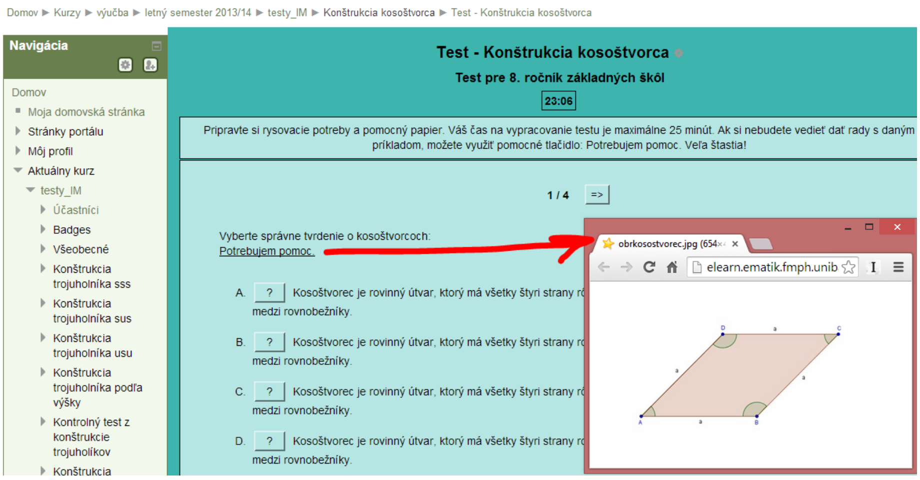

Figure 8 A sample use of HotPotatoes.

Conclusion

Experience from the past years has shown, that for teachers and future teachers of mathematics, the content we provide in our course is very useful and so is its e-learning form. In the State educational program (Mathematics, ISCED 3A), it is recommended to use other forms of education, such as in a computer classroom. Pupils should learn to use ICT for searching, processing, saving and presenting information, that should make difficult calculations and procedures easier for them. The correct and effective use of digital technologies is undoubtedly beneficial for teaching mathematics. Because of these facts, it seems this continuing education of teachers of mathematics for raising their professional competences to use digital technologies, is reasonable.

Marc Prensky defined four degrees of introducing digital technologies into education – typically a four-step process: 1. Dabbling, 2. Doing old things in old ways, 3. Doing old things in new ways, 4. Doing new things in new ways (Prensky, 2005).

At the majority of Slovak schools, the situation is somewhere at degree 1 or 2. An effective introduction of digital technologies, however, requires at least a level 3, which means that teachers must learn new educational methods and forms and possibly even to innovate the content. Our courses are oriented precisely on improving the competencies of teachers, so that they can get to level 3 or 4, which will improve the effectiveness of teaching mathematics.

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Acknowledgement

This paper arose thanks to the support of the project KEGA 094UK-4/2013 „Ematik+, Continuing education of mathematics teachers“, http://elearn.ematik.fmph.uniba.sk.

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Rediscovered Anaglyph in Program GeoGebra

Mária Kmeťová*
Department of mathematics, Faculty of Natural Sciences, Constantine the Philosopher University in Nitra, Tr. A. Hlinku 1, SK - 949 74 Nitra

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Abstract

In this paper we study several applications of geometry beginning with visualization of spatial object in the plane up to modelling of special curves and surfaces in computer graphics. We show the usefulness of the dynamic program GeoGebra for better visualization of geometric relationships in space using anaglyphs.

Keywords: Anaglyph, dynamic geometric program, GeoGebra, Monge projection, curves and surfaces in CAGD.

Classification: 97G80

Introduction

Many people say that they are not very good in space vision or that they have poor spatial sense. The typical belief is that the child is either born with spatial sense or not. But we know that rich experiences with shape and spatial relationships (consistently provided) can develop the spatial sense to a large extent.

Dynamic geometric program GeoGebra [1] is a very useful helpmate for rigorous visualization of geometric relations in plane and in space. For visualization the shapes in the space we can use two types of outputs. The first one is the widely used parallel projection of the given 3D object (can be changed to central projection, but in this case the parallel lines project into nonparallel lines and the picture could be not suitable for teaching basic geometric properties or relationships). The second and the newest possibility for visualization of spatial objects in GeoGebra is to choose anaglyph. The program creates two central projection images into one picture in red and cyan colour. Using anaglyph filter glasses we can see the spatial object in space partly in front of and partly behind the screen. This is a perfect tool for training spatial geometry abilities, mainly for students with lower spatial sense.

Anaglyph

Anaglyphic stereogram (anaglyph) is the name given to the spatial effect achieved with encoding each eye`s image using chromatically opposite (usually red and cyan, occasionally red and green in Europe) filters. “Anaglyph” is derived from two Greek words meaning “again” and “sculpture”. (So we again find ourselves discovering this technique in the 21st century.)

*Corresponding author: mkmetova@ukf.sk
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Anaglyph images contain two differently filtered coloured images, one for each eye. The images are created from two central projection images projected from the centres approximately 6.5 centimetres apart, which is the centre distance typically between human eyes. The central projection of the object projected from the left centre is green/cyan coloured, the image projected from the right centre is red coloured. Viewing an anaglyph stereo image is then easy: we need a red filter over the left eye and a green/cyan filter over the right eye. The eye that is covered with the red lens will see only the green/cyan image in black. Similarly, the eye that is covered with the green/cyan lens will see only the red image in black. If colours on the glasses fit with the colours on the printed or displayed material, we must not see the red image through the red glass with left eye, respective the green image through the green glass with right eye. Each eye will see only the image prepared for it (see Figure 1) and so the perception of depth will be created.

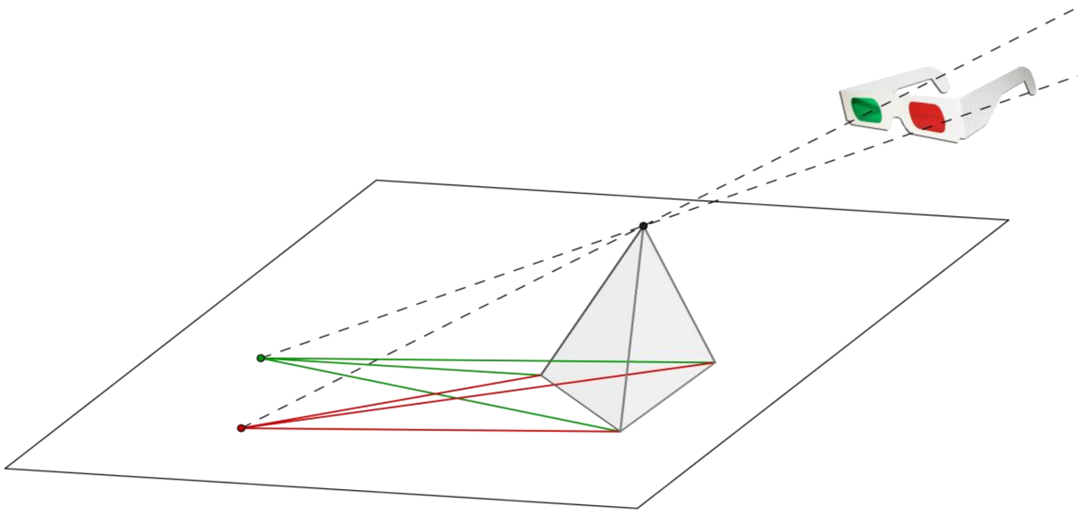

Figure 1

The first method to produce anaglyph images was developed in 1852 by Wilhelm Rollman in Leipzig, Germany. In 1858, a French gentleman named Joseph D'Almeida used this technique to project anaglyphic stereo lantern slides onto a theatre screen [2]. The audience viewing the exhibition was adorned with red and green goggles to witness the very first 3D slide show ever.

William Friese-Green created the first 3D anaglyphic motion pictures in 1889, using a camera with two lenses, which were first shown to the public in 1893. Anaglyphic films called “plastigrams” enjoyed great popularity in the 1920s. These used a single film with a green image emulsion on one side of the film and a red image emulsion on the other. In 1922, an interactive plastigram, entitled “Movies of the Future,” opened at the Rivoli Theater in New York [2]. Anaglyphic images have been used in comic books, newspapers, and magazine ads. In 1953, 3D comic books were invented and distributed with red and green “space goggles.” But anaglyphs were used not only for entertainment. Some descriptive geometry textbooks with anaglyph illustration also arose. One of them is the textbook [3]; its first edition is from the year 1959.

Anaglyphs are used today in science for instance to map the topography and geology of the planets and moons. Stereo images can reveal geologic features of the planet not otherwise visible. Just as the early stereogram images allowed people around the world to experience different places and events, stereo images of the planets help us to experience these different worlds in a more tangible way. Scientists at the Jet Propulsion Laboratory used anaglyphic images to look at the surface of Mars using pictures sent back by the Pathfinder spacecraft. This allowed the scientists to experience the planet in a more familiar way. The features of the surface and the rocks could be analysed in more detail using these images. The Viking spacecraft that visited Mars in the 1970s also returned some images in stereo, as did the Apollo missions to the Moon in the 1960s (Figure 3) and 1970s [2]. Figure 2 shows the image presented by NASA, a two-colour anaglyph from the Mars Pathfinder mission (red-cyan glasses are recommended to view the images correctly).

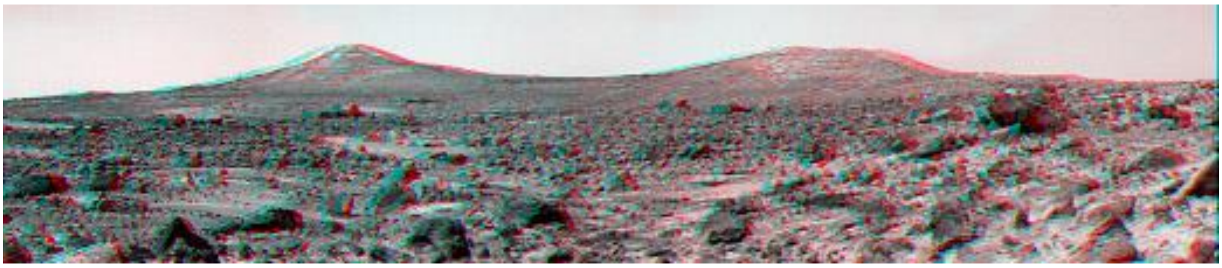

Figure 2

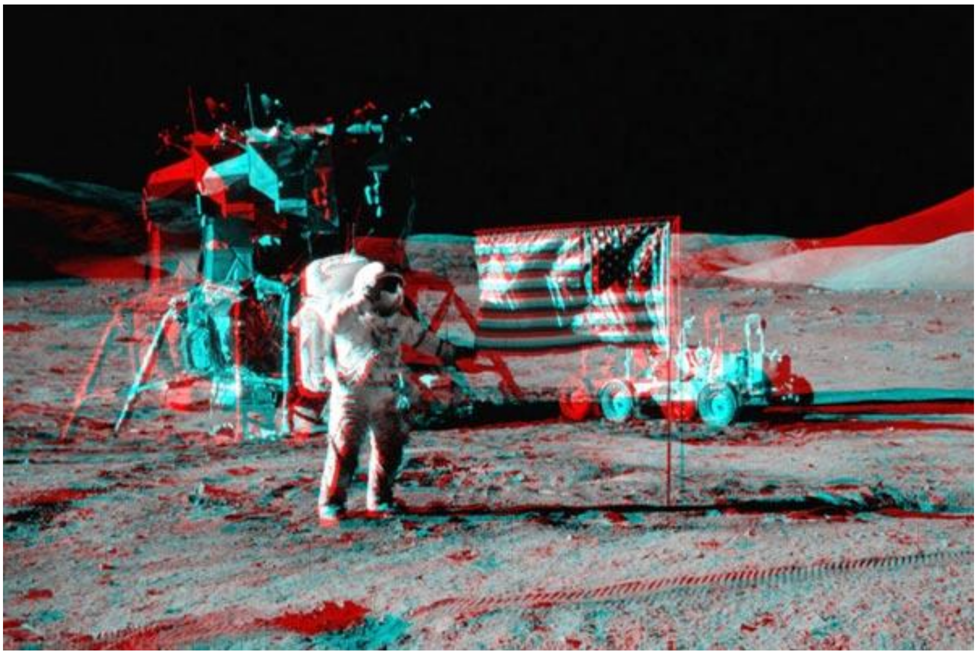

Figure 3

Using anaglyph in dynamic program GeoGebra

Monge projection is a short name for orthogonal projections to two perpendicular (horizontal and vertical) planes.

The double-view orthogonal projection is named after Gaspard Monge, the French mathematician, the inventor of descriptive geometry. Monge projection has a wide range of uses in technical drawing and computer graphics. This is the reason, why Monge projection forms a basic part of the subject Constructive Geometry in teacher training study. Despite the fact that the program GeoGebra does not support Monge projection, it is possible to display a front view and a top view of the objects quite successfully. As an example we show the solution of an intersection problem from the university textbook [4]. The task requires determining the intersection of two triangles *ABC* and *MNP* in Monge projection. The coordinates of the vertices are given as *A*[-1, 0, 6], *B*[-4.5, 4.5, 0], *C*[2, 7, 1.5], *M*[0.5, 1, 0.5], *N*[3.5, 3, 4.5], *P*[-2.5, 7, 5].

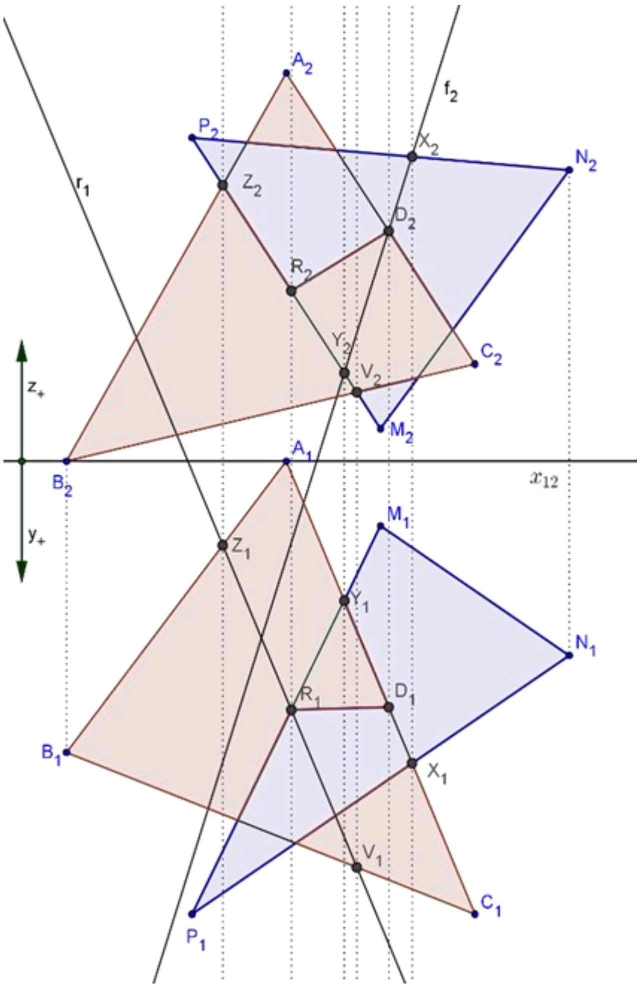

Figure 4

The *x* axis of the plane coordinate system in the graphics screen of GeoGebra is used as an x_{12} axis in Monge-projection. That means that the top view of the triangle *ABC* is given by (see Figure 4) A_1[-1, 0], B_1[-4.5, -4.5], C_1[2, -7], and the front view is given by A_2[-1, 6], B_2[-4.5, 0], C_2[2, 1.5].

The top view and the front view of the triangle *MNP* are given similarly. In general, the top view of a space point *X*[*a*, *b*, *c*] is a plane point X_1[*a*, -*b*], and the front view is given by X_2[*a*, *c*]. The further solution of the task is identical with the solution in paper form. The advantage of this dynamical geometry form of solution is that the coordinates of basic points are variable. We can create a new task from the previous one and confront the solutions. This variability has only an assumption that the front view and the top view of each point move on a perpendicular line to the x_{12} axis.

A problem appears, when some students are not able to imagine these objects in 3D space.

Program GeoGebra can be helpful in this case too. We can use the 3D version and display the objects in space (see left part of the figure 5).

To increase the perception of depth and to have full stereo image, we can use the newest function of the program, which divides the 3D picture into two central projections in green and red colours, i.e. creates anaglyph of the original picture. The right side of the figure 5 shows the anaglyph of the left side picture (red-cyan glasses are recommended to view the image correctly). After using the rotation function of the program and view the image through red-cyan glasses, we receive a full spatial experience with rotating black-and-white objects in front of and behind the screen.

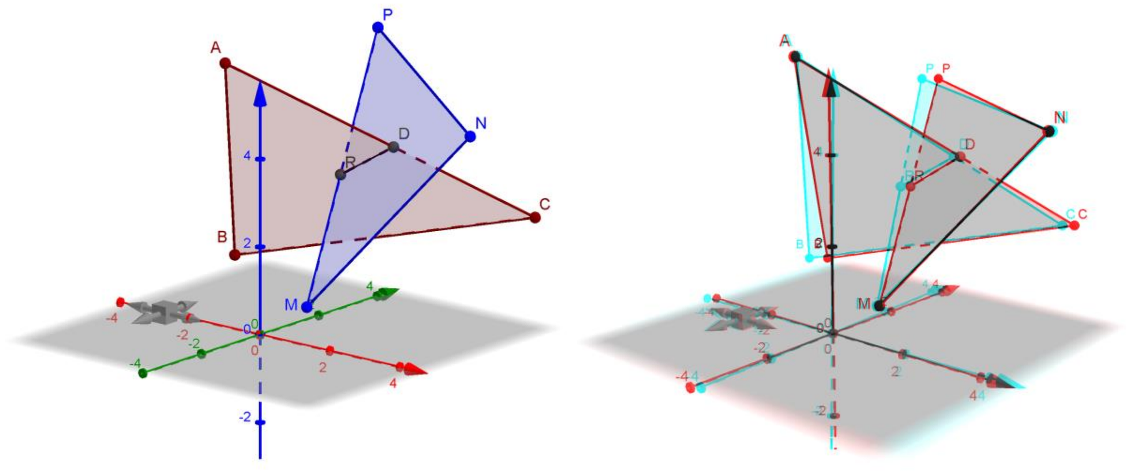

Figure 5

The second ideal usage of anaglyphs in geometry teaching appears in the subject Curves and Surfaces [5] for computer graphics study. Anaglyph is a great support for students with lower level of space vision. As an example we show the bilinear surface with parabola diagonals. Students are able to compute the equation of the curve on the surface but often, they are not able to imagine it in space. Figure 6 shows the situation in parallel projection, and figure 7 shows the same situation in space (of course, red-cyan glasses are recommended to view the image correctly). In the pictures, the control points of the diagonal curves appear. To increase the spatial experience, we can choose a rotation function of the program.

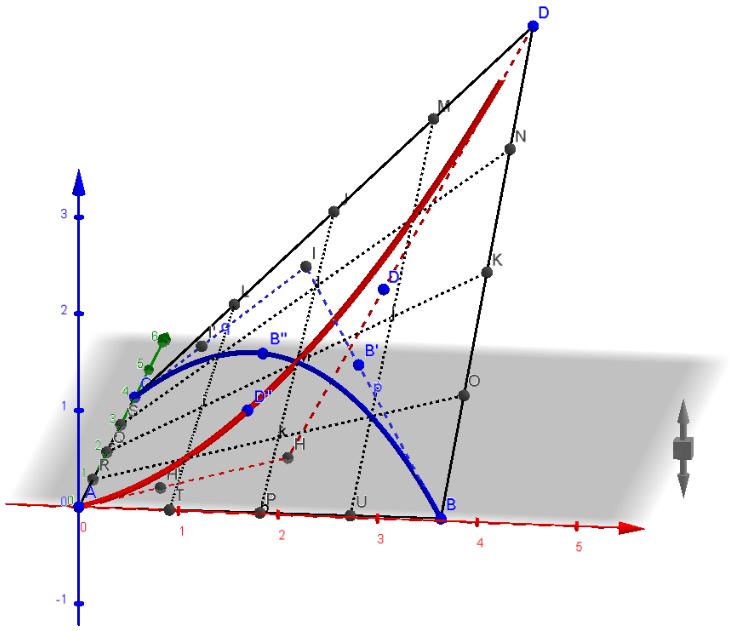

Figure 6

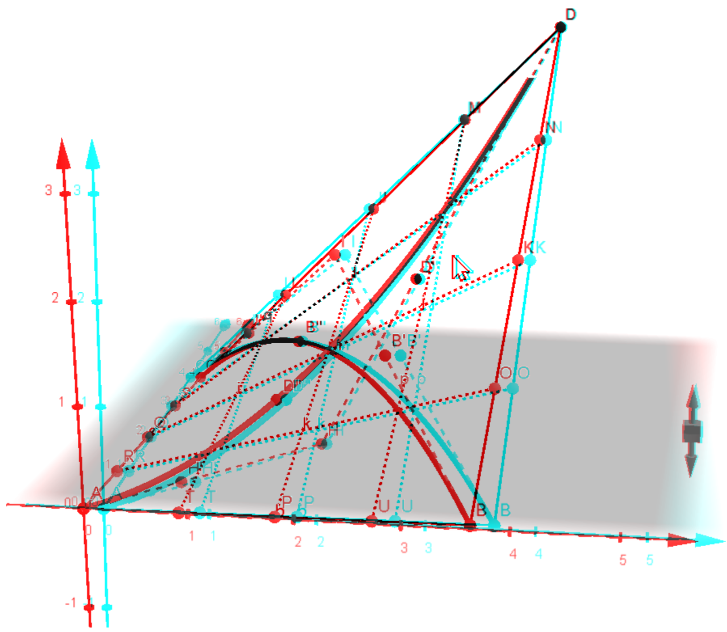

Figure 7

Conclusion

The aim of the paper was to demonstrate the usefulness of anaglyph in the program GeoGebra for teaching spatial geometry relationships. At first we dealt with the history of anaglyph, than we discussed, how a perfect 3D experience was created. We have showed the usefulness of this feature of the program GeoGebra on the seminars of geometry in subjects Monge projection and Curves and Surfaces.

Acknowledgement

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Problem Posing as a Tool for Developing and Designing Tasks

Lukáš Lednický*

Department of mathematics, Faculty of Natural Sciences, Constantine the Philosopher University in Nitra, Tr. A. Hlinku 1, SK - 949 74 Nitra

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Abstract

Problem posing is mostly applied during a lesson in such way, that the teacher prepares a material, situation or a problem and the students pose some new problems and solve them. It is possible to use problem posing outside of the classroom. We want to show that problem posing can be successfully used for developing and designing new tasks and problems for mathematics teaching and learning. We describe the What-If-Not strategy of problem posing and give an example of creating new problems using this strategy. We also solve some of the newly created problems.

Keywords: Problem posing, problem solving, task design.

Classification: D50

Introduction

The tasks teachers pose in their classrooms deserve important consideration because they open or close the students' opportunity for meaningful mathematics learning. They convey implicit messages about the nature of mathematics: what it is, what it entails, and what is worth knowing and doing in mathematics (Crespo, 2003). This is a responsibility not only of the teachers but also authors of textbooks and task designers, because teachers pose problems that come from textbooks or other literature. Now a question arises, where do the problems in a textbook come from? In most cases, they are a product of a creative thinking of the authors or anyone, who prepares a new problems for teaching and learning. We assume that adopting principles and strategies of problem posing can help them in generating original problems.

Problem-posing

Problem posing involves generating of new problems and questions aimed at exploring a given situation as well as the reformulation of a problem during the process of solving it (Silver, 1994). In the first case, problem posing is a divergent task that has multiple possible answers (posed problems). Therefore, problem posing is considered to be a creative generation task that requires productive thinking. To promote diverse and flexible thinking, it is critical for learners to generate diverse problems. However, it has been confirmed that problems generated by novice learners lack diversity (Kojima & Miwa, 2008). Advocates for problem posing typically argue that experience with mathematical problem posing can promote students' engagement in authentic mathematical activity; allow them to encounter

*Corresponding author: lukas.lednicky@ukf.sk
DOI: 10.17846/AMN.2015.1.1.92-99

many problems, methods and solutions, and promote students' creativity – a disposition to look for new problems, alternate methods, and novel solutions (Singer et al, 2011).

A systematic training focused on problem transposition using various representations, problem extension by adding new operations or conditions, comparison of various problems in order to assess similarities and differences, or analysis of incomplete or redundant problems can raise students' awareness of meaningful problems (Singer et al, 2011). Effective teaching should focus on representational change, within a variety of activities, which specifically address students' motor, visual, and verbal skills, as well as transfers in between them (Singer et al, 2011). In problem-posing contexts, students are stimulated to make observations, experiment through varying some data and analyzing the results, and devise their own new problems that could be solved by equally using similar or different patterns. The processes by which students continue given series or patterns provide information about the cognitive approaches they use in problem solving (Singer, Voica, 2008)

If problem posing is such an important intellectual activity, the first question we need to ask is who can pose mathematical problems. One important line of research in problem posing has been exploring what problems teachers and students can pose (Singer et al, 2011). The process of posing problems in this line proceeds during a mathematics lesson, where a problem situation is given and students are asked to pose new problems. The process of problem posing can be also used out of the lesson by teachers, authors of textbooks or task designers.

What-If-Not strategy

The What-If-Not (WIN) strategy was introduced by Brown and Walter (2005). It is based on the idea that modifying the attributes of a given problem could yield new and intriguing problems which eventually may result in some interesting investigations. The strategy cen be described in following stages:

Level 0 – Choosing a Starting Point: As a starting point for problem posing can be used a concrete material, existing problem or a theorem.

Level 1 – Listing Attributes: Look at what is given and find the important attributes of the starting point.

Level 2 – What-If-Not-ing: Choose one or more attributes from the list and ask "What if not this attribute?" and list some alternatives for this attribute.

Level 3 – Question Asking or Problem Posing: Choose an alternative to the attribute and pose a question or problem.

Level 4 – Analyzing the Problem: Analyzing and trying to answer the question gives us a deeper insight into the problem.

The process of the WIN strategy seems to be linear, but it is cyclical in fact. Brown and Walter (2005) explain: "Our scheme, however, is not as linear as it may seem from this list. Almost every part can (and does) affect others. A new question may trigger a new attribute, and a new attribute may in turn trigger a new question (for example). This in turn may enable you to see the original phenomenon in a new light."

Using problem-posing for task design

Many projects or final thesis at universities have a similar objective, to create a collection of new task for a certain part of mathematics or a specific type of tasks. Strategies of problem-posing can be used to fulfil this objective. Authors often unwittingly and intuitively use the WIN strategy to create new tasks. We suggest that intentional usage of problem-posing strategies can increase quantity and quality of designed tasks.

Now we show how to develop new problems using the WIN strategy. As a starting point we choose a problem we found in (Kopka, 2010). In the following problems, we call a square with a side of length k a square of size $k \times k$ and similarly a rectangle of size $k \times l$ is a rectangle with sides of length *k* and *l*. An equilateral triangle with the side of length *k* is called a triangle of size *k*.

Problem: Determine the number of all squares in a square grid of size $n \times n$, where *n* is a natural number other then 0. (Kopka, 2010, p.124)

Now we need to create a list of attributes of this problem. In other word we can describe the situation that we have a square with side of length *n* divided into squares with side of length 1 and we want to know the number of all squares in the grid. In the previous sentence we underlined the most important attributes of the problem. Now we can start what-if-not-ing. We write down some possible alternatives for each attribute:

Square of size $n \times n$ – triangle of size *n*, rectangle of size $m \times n$, square of size $n \times n$ with a missing square in a corner(s).

Squares of size 1×1- triangles of size 1, squares of different sizes and their combination.

Number of all squares – number of squares of certain size, number of all triangles, number of all rectangles.

Choosing some of the alternatives we can pose the following problems:

Problem 1: Count the number of all triangles in a triangular grid of size *n*.

Problem 2: Count the number of all rectangles in a square grid of size $m \times n$.

Problem 3: Count the number of all squares in a square grid of size $n \times n$ divided into one square of size 2×2 and squares of size 1×1 (see Figure 1, left).

Problem 4: Count the number of all squares in a square grid of size $n \times n$ divided into one square of size 2×2 and squares of size 1×1, where the square of size 2×2 is placed arbitrarly in the grid..

Problem 5. Count the number of all squares in a square grid of size $n \times n$ with one square missing in the up-left corner (see Figure 1, right).

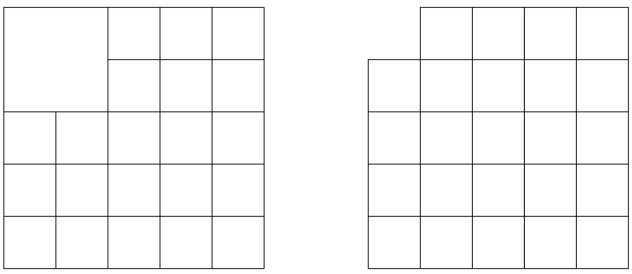

Figure 1

It is possible to pose even more new problems by varying the attributes of the original problem. Of course replacing one attribute may lead to a trivial problem so it needs to also replace another attribute too. For example if we replace the number of all squares by triangles, the new problem has no solution.

We assume that the stated problems fully present the potential of WIN strategy for generating new problems. As a last step of the WIN strategy, we analyse and solve four of these problems. In the solutions, we use the following well known identities for finite sums.

$$\sum_{k=1}^{n} k = \frac{1}{2}n(n+1), \sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1), \sum_{k=1}^{n} k(k+1) = \frac{1}{3}n(n+1)(n+2).$$

Problem 1: Count the number of all triangles in a triangular grid of size *n*.

Solution: In such a triangle there are triangles of different sizes and some are rotated about 180 degrees. We will try to count all these types of triangles separately. First we focus on the triangles in the basic position. We start with triangles of size 1. We can notice that in the first row there is only 1 such triangle. In the second row there are 2. In the i-th row there are *I* triangles of size 1. So the number of triangles of size 1 is $1+2+3+\cdots+n$, which we denote by T_n, the triangular number. Now we will count the triangles of size 2, by moving the triangle in all the possible positions. We will track the position of the upper triangle of size 1 in the triangle of size 2 (see Figure 2). The tracked triangle occupies the positions of a triangle of size 1 in the first $n-1$ rows. Therefore the number of triangles of size 2 is T_{n-1}. This idea can be generalized for any triangle of size *I* giving the result T_{n-i+1}.

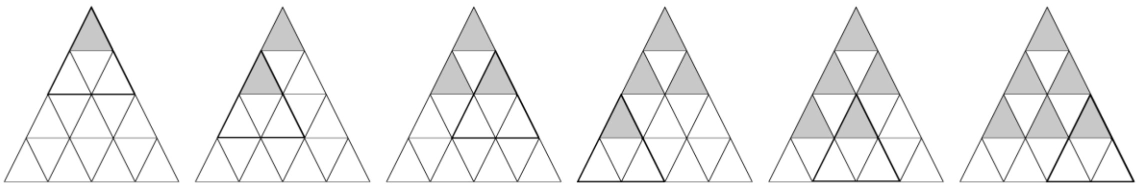

Figure 2

Now we will focus on the rotated triangles. First of all we need to realize that not all sizes are possible for the rotated triangles. The side of a rotated triangle cannot be longer then $\frac{n}{2}$. If *n* is odd the largest size is equal to $\frac{n-1}{2}$, for even *n* it is $\frac{n}{2}$. We can write it jointly $\left\lfloor\frac{n}{2}\right\rfloor$. Now we can count the triangles similarly as in the case of triangles in the basic position. The number of triangles of size 1 is in row *I* equal to $i-1$. So summing it up gives T_{n-1}. For triangles of size 2 we will track the position of the triangle of size 1 in the basic position in the middle. It is obvious that the tracked triangle will not occupy any position in the first two rows and in the last row. All the possible positions of the tracked triangle are the triangles of size 1 in the basic position in a triangle of size $n-3$. The number of rotated triangles of size 2 is equal T_{n-3}. We can generalize this for triangles of size *i*, where *I* is a natural number satisfying inequality $1 \le i \le \left\lfloor\frac{n}{2}\right\rfloor$.

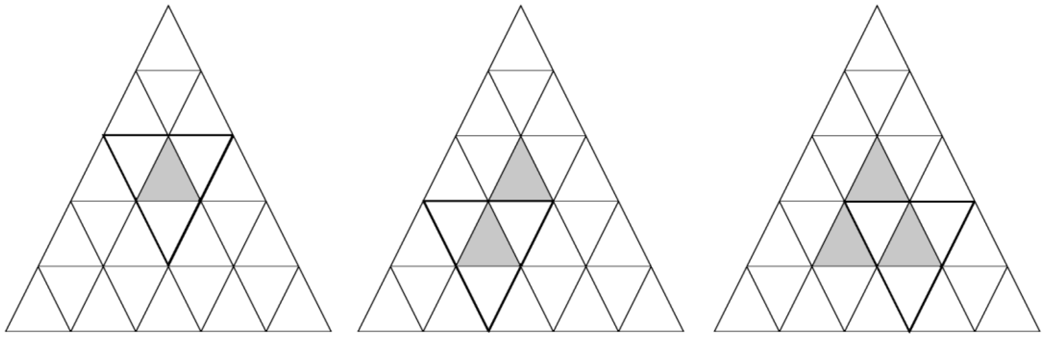

Figure 3

We summarize our findings in Table 1.

Table 1

Size		1	2	3	4	…	$n-1$	n
Position	Basic	T_n	T_{n-1}	T_{n-2}	T_{n-3}	…	T_2	T_1
	Rotated	T_{n-1}	T_{n-3}	T_{n-5}	T_{n-7}	…	0	0

We want to derive an explicit formula for counting the number of triangles. First we count the number of triangles in the basic position. We know that $T_n = \frac{1}{2}n(n+1)$. Then

$$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n} \frac{k(k+1)}{2} = \frac{1}{2}\sum_{k=1}^{n} k(k+1) = \frac{1}{6}n(n+1)(n+2).$$

We need to distinguish two cases for counting the rotated triangles. If *n* is even, we sum up triangular numbers with odd index.

$$\sum_{k=1}^{n} T_{2k-1} = \sum_{k=1}^{n} \frac{2k(2k-1)}{2} = \sum_{k=1}^{n} k(2k-1) = 2\sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} k =$$

$$= \frac{1}{3}n(n+1)(2n+1) - \frac{1}{2}n(n+1) = \frac{1}{6}n(n+1)(4n-1).$$

If *n* is odd, we sum up triangular numbers with even index.

$$\sum_{k=1}^{n} T_{2k} = \sum_{k=1}^{n} \frac{2k(2k+1)}{2} = \sum_{k=1}^{n} k(2k+1) = 2\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k =$$

$$= \frac{1}{3}n(n+1)(2n+1) + \frac{1}{2}n(n+1) = \frac{1}{6}n(n+1)(4n+5).$$

We denote by $t(n)$ the number of all triangles in the triangular grid of size *n*. For *n* even we have:

$$t(n) = \sum_{k=1}^{n} T_k + \sum_{k=1}^{n/2} T_{2k-1} = \frac{1}{6}n(n+1)(n+2) + \frac{1}{6}\frac{n}{2}\left(\frac{n}{2}+1\right)\left(4\frac{n}{2}-1\right) =$$
$$= \frac{1}{8}n(n+2)(2n+1).$$

For *n* odd we have:

$$t(n) = \sum_{k=1}^{n} T_k + \sum_{k=1}^{(n-1)/2} T_{2k} = \frac{1}{6}n(n+1)(n+2) + \frac{1}{6}\frac{n-1}{2}\left(\frac{n-1}{2}+1\right)\left(4\frac{n-1}{2}+5\right) =$$
$$= \frac{1}{8}(n+1)(2n^2+3n-1).$$

For example we can count $t(4)$ and $t(5)$.

$$t(4) = \frac{1}{8}4.6.9 = 27, \qquad t(5) = \frac{1}{8}6.64 = 48.$$

Problem 2: Count the number of all rectangles in a square grid of size $m \times n$.

Solution: We can find rectangles of different sizes in the rectangle. Some of them are squares. We will distinguish between rectangles of size $a \times b$ and $b \times a$. If we have a rectangle of size $a \times b$, where $1 \le a \le m$ and $1 \le b \le n$, we can place it in the rectangle in such way that their top left corners merge into one. This is one possible way of placing the rectangle of $a \times b$. Now we can move it to the right side into one of the $m - a$ positions. This gives us $m - a + 1$ positions in the horizontal line. Similarly we can move the rectangle in the vertical line. There we obtain $n - b + 1$ positions. The number of positions of the rectangle of size $a \times b$ in a rectangle of size $m \times n$ is $(m - a + 1)(n - b + 1)$. Now we put together a table in which each cell contains the number of rectangles of size $a \times b$ (see Table 2). The last column contains sums of rows.

Table 2

Size	1	2	…	$m-1$	m	Sum
1	mn	$(m-1)n$	…	$2n$	n	$n(1+2+\cdots+m)$
2	$m(n-1)$	$(m-1)(n-1)$	…	$2(n-1)$	$n-1$	$(n-1)(1+2+\cdots+m)$
⋮	⋮	⋮		⋮	⋮	⋮
$n-1$	$2m$	$2(m-1)$	…	4	2	$2(1+2+\cdots+m)$
n	m	$(m-1)$	…	2	1	$(1+2+\cdots+m)$

We denote by $r(m,n)$ the number of all rectangles. We need to sum up all the expressions in the last column of Table 2.

$$r(m,n) = m(1+2+\cdots+n) + (m-1)(1+2+\cdots+n) + \cdots + (1+2+\cdots+n) =$$
$$= (1+2+\cdots+m)(1+2+\cdots+n).$$

We can also derive a recurrence for $r(m,n)$. If we take a look on Table 3, we can see that it contains same expressions as Table 2 except the first row and first column.

We can write:

$$r(m+1,n+1)=r(m,n)+(n+1)\big(1+2+\ldots+(m+1)\big)+(m+1)(1+2+\cdots+n)=$$
$$=r(m,n)+\frac{1}{2}(n+1)(m+1)(m+2)+\frac{1}{2}(m+1)n(n+1)=$$
$$=r(m,n)+\frac{1}{2}(m+1)(n+1)(m+n+2).$$

We need to determine $r(1,n)$ and $r(m,1)$. If we look into Table 2 from column *m* and row *n* we obtain:

$$r(m,1)=1+2+\cdots+m=\frac{1}{2}m(m+1).$$
$$r(1,n)=1+2+\cdots+n=\frac{1}{2}n(n+1).$$

Table 3

	1	2	3	...	m	$m+1$
1	$(m+1)(n+1)$	$m(n+1)$	$(m-1)(n+1)$		$2(n+1)$	$(n+1)$
2	$(m+1)n$	mn	$(m-1)n$	...	$2n$	n
3	$(m+1)(n-1)$	$m(n-1)$	$(m-1)(n-1)$	...	$2(n-1)$	$n-1$
⋮	⋮	⋮	⋮		⋮	⋮
n	$2(m+1)$	$2m$	$2(m-1)$	...	4	2
$n+1$	$(m+1)$	m	$(m-1)$	...	2	1

For example we can count $r(5,3)$. By the explicit formula we have:

$$r(5,3)=(1+2+3+4+5)(1+2+3)=15.6=90.$$

Using the recurrence we obtain:

$$r(5,3)=r(4,2)+\frac{1}{2}5.3.8=r(3,1)+\frac{1}{2}4.2.6+60=6+24+60=90.$$

Remark: If we choose $m=n$, we have a similar problem of determining the number of rectangles in a square of size $n\times n$. Solution of this problem is the same, but the derived formulas have the form:

$$r(n)=(1+2+\cdots+n)^2=\sum_{i=1}^{n}i^3=\frac{1}{4}n^2(n+1)^2$$
$$r(n+1)=r(n)+(n+1)^3, r(1)=1.$$

In problems 3 and 5 we use the result of the original problem of counting the number of all squares in a square grid. We denote it by $s(n)$. It can be derived even from table 2 looking at the cells laying at the main diagonal and setting $m=n$. We see that $s(n)=\sum_{k=1}^{n}k^2$.

<u>Problem 3</u>: Count the number of all squares in a square grid of size $n\times n$ divided into one square of size 2×2 and squares of size 1×1 (see Figure 1, left).

Solution: We will count, how many squares are missing in this kind of grid. There are missing 4 squares of size 1×1. The square of size $n \times n$ is still in the grid. As for the squares of other sizes, there are always 3 missing squares of each size. Together it is $3n - 2$ missing squares. If we denote $s_1(n)$ the number of all squares in this kind of a square grid, then:

$$s_1(n) = s(n) - (3n - 2) = \frac{1}{6}n(n+1)(2n+1) - (3n-2) = \frac{1}{6}(n-1)(n+4)(2n-3).$$

Problem 5. Count the number of all squares in a square grid of size $n \times n$ with one square missing in the up-left corner (see Figure 1, right).

Solution: We will count how many squares disappear by removing the square from the corner. All squares in a square grid that are placed in such way that they cover the removed square, do not appear in the grid any more. Number of these squares is exactly *n*, because there is only one square of each size that can be placed in a square grid in the described way. If we denote $s_2(n)$ the number of squares in the square grid without a square in the corner, then:

$$s_2(n) = s(n) - n = \frac{1}{6}n(n+1)(2n+1) - n = \frac{1}{6}n(n-1)(2n+5).$$

Conclusion

We think that problem posing and especially the WIN strategy can be a very useful for developing new original tasks and problem for mathematics teaching and learning. We showed an example of using the WIN strategy and formulated five new problems of which we solved four. We could pose more problems, but the stated problems fulfil our objective to point out a possible use of problem posing.

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Testing of the Fundamental Mathematical Knowledge

Zuzana Malacká*

Department of mathematics, Faculty of humanities University of Zilina,
Univerzitná 8215/1, SK - 010 26 Žilina

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Abstract

The study results of mathematics at a university are determined by knowledge of secondary curriculum. The content and the scope of teaching of mathematics are reducing at secondary schools and the graduation exam in mathematics is not compulsory. In the paper we present on understanding of fundamental mathematical knowledge in the first study year of bachelor study at the Faculty of Civil Engineering University of Žilina in Žilina . Also there are listed the most common deficiencies and suggestions to improve the situation.

Keywords: Mathematical knowledge, pedagogical experiment, research, study results, mathematical thinking .

Classification: B10, B40

Introduction

Mathematics is an activity concerned with logical thinking, spotting patterns, posing premises and investigating their implications and consequences. It also involves the study of the properties of numbers and shapes, the relationship between numbers, inductive and deductive thinking and the formulation of generalizations. Mathematics is a creation of the human mind and therefore becomes primarily a way of thinking thus facilitating problem solving.

Mathematics is very important object in modern society, so mathematics education must to satisfy the needs of society. In fact, mathematics education is always behind social development. In this system, our children and students, the new masters of their own learning, are asked to somehow discover the ways of arithmetic by trying to figure out worded math problems. Today math isn´t only about numbers, it´s about words and theories, as if the curriculum was written by folks, who hate the clear logic of pure mathematics. Mathematics education must never stop reforming. It can become a power which pushes society forward [2].

Slovakia is a developed country. Mathematics education is taken into account by government and education departments. They support and help mathematics education reform. Where is mathematics education reform from? From technology? From textbooks? Or from the mathematics education system?

*Corresponding author: zuzana.malacka@fpv.uniza.sk
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Research setting

I have taught mathematics on the university level for 19 years. Students have been cheated by not having been taught the basic fundamental skills that are essential to an understanding of the subject. Memorizing multiplication tables and mastering long divisions are as fundamental to math as learning the alphabet is reading and writing. Many of our students are ill prepared for higher level mathematics. Students need to memorize basic mathematics facts in order to focus on higher- level concepts. As a teacher I know that give some examples of what you believe is wrong with the curriculum. This curriculum taught without an experienced teacher mitigating its effects by insisting on mastery of fundamentals results in the majority of students becoming frustrated and losing their math confidence entirely.

For measuring the high school mathematics knowledge of the students I made the first year students to write a test, according to that we can say that in the previous knowledge of many our students there are big differences. For the purpose of experiment I randomly chose the group 85 students from the 1-st grade of Faculty of Civil Engineering in Žilina.

Test

1. Which of the following numbers are natural, integer, rational or irrational numbers?
-2; $-\frac{3}{2}$; $\frac{30}{15}$; $\frac{4}{12}$; 5,07; 9; -81,5; $4,\overline{1}$; π =3,14159…; $\sqrt{3}$; $-\sqrt{7}$; $\sqrt[3]{8}$; $\sqrt[5]{3}$; $\sqrt[3]{-64}$.
Answers: correct **27** partially correct **32** incorrect **26**

Analyzing of some solutions of students:
Some students correctly identified numbers but they could not designate sets of numbers. The biggest problem for many students was to determine rational and irrational numbers.

2. Edit the expressions

a) $(\sqrt{3}-2x)\,(\sqrt{3}+2x)$

b) $\sqrt{\frac{9}{4}x^2}$

c) $\sqrt[3]{(1-x)^3}$

Answers: correct **33** partially correct **31** incorrect **21**

Analyzing of some solutions of students:
Some students didn´t know the formula

a) $(\sqrt{3}-2x)\,(\sqrt{3}+2x) = 3-2\sqrt{3}x+4x$

b) $\sqrt{\frac{9}{4}x^2} = \frac{3}{2}\sqrt{x^2}$

c) $\sqrt[3]{(1-x)^3} = (1-x)^2$

3. Edit the complex fractions

a) $\dfrac{2-\frac{x^2+y^2}{xy}}{\frac{x}{y^2}-\frac{2}{y}+\frac{1}{x}}$ b) $\left(\frac{u+v}{u-v}+\frac{u-v}{u+v}\right):\left(\frac{u+v}{u-v}-\frac{u-v}{u+v}\right)$ c) $\left(\frac{2ab+a}{a+2bc}\right):\left(1-\frac{c}{a}\right)$

Answers: correct **19** partially correct **24** incorrect **52**

Analyzing of some solutions of students:

The students made wrong steps in operations with fractions.

They made frequently mistakes when they shortcut the fractions and they often forget to write a conditions.

a) $\frac{2-\frac{x^2+y^2}{xy}}{\frac{x}{y^2}-\frac{2}{y}+\frac{1}{x}} = \frac{2-\frac{x^2+y^2}{xy}}{\frac{xyx-2y^2x+y^2y}{y^2yx}} = \frac{2-x^2+y^2}{\frac{x-2y^2x+y^2y}{y^2}} = \frac{2-x^2+y^2}{x-2x+y} = \frac{2-x^2+y^2}{-x+y}$

b) $\left(\frac{u+v}{u-v}+\frac{u-v}{u+v}\right):\left(\frac{u+v}{u-v}-\frac{u-v}{u+v}\right) = \left(\frac{u+v+u-v}{u-v.u+v}\right):\left(\frac{u+v-u-v}{u-v.u+v}\right) = \left(\frac{2u}{u-v.u+v}\right):\left(\frac{0}{u-v.u+v}\right) = 0$

c) $\left(\frac{2ab+a}{a+2bc}\right):\left(1-\frac{c}{a}\right) = \left(\frac{2b+1}{1+2bc}\right):\left(\frac{1-c}{a}\right) = \left(\frac{2b+1}{1+2bc}\right).\left(\frac{a}{1-c}\right) = \frac{2b+1.a}{1+2bc.1-c}$

4. Solve equations using the decomposition

a) $2x^2 - 8 = 0$

b) $x^2 + 3 = 0$

c) $x^2 - 5x + 4 = 0$

Answers: correct **17** partially correct **40** incorrect **28**

Analyzing of some solutions of students:

a)
$$2x^2 = 8$$
$$x^2 = 4$$
$$x = 2$$

b)
$$x^2 + 3 = 0$$
$$x^2 + 3 = (x + \sqrt{3})^2$$
$$x = -\sqrt{3}$$

c)
$$x^2 - 5x + 4 = 0$$
$$x(x-5) + 4 = 0$$
$$x(x-5) = -4$$
$$x = 2,\ x = 3$$

5. In a class, the ratio of boys to girls is $\frac{4}{5}$. If there are 12 boys in the class, how many girls are there?

Answers: correct **29** partially correct **0** incorrect **56**

Analyzing of some solutions of students:

The students solved this example right or they had the problem to express the text mathematically and they didn´t solve it.

6. Which numbers are Pythagorean triples? Write yes or no.

a) 3, 4, 5

b) 4, 5, 6

c) 24, 45, 51

Answers: correct **36** partially correct **0** incorrect **49**

Analyzing of some solutions of students:

$$c^2 = a^2 + b^2$$

a) 3, 4, 5 yes

$$5^2 = 3^2 + 4^2$$
$$25 = 9 + 16$$
$$25 = 25$$

b) 4, 5, 6 no

$$6^2 = 4^2 + 5^2$$

$$36 = 16 + 25$$

$$36 \neq 41$$

c) 24, 45, 51 yes

$$51^2 = 24^2 + 45^2$$

$$2601 = 576 + 2025$$

$$2601 = 2601$$

Some students could not use Pythagoras' Theorem. Even many of them did not know it. They only guessed or used triangle inequality.

a) 3, 4, 5 yes $3 + 4 > 5$

b) 4, 5, 6 yes $4 + 5 > 6$

c) 24, 45, 51 yes $24 + 45 > 51$

7. Draw the graphs of functions

a) $y = 2x - 3$ b) $y = 7x$

c) $y = -3x + 5$ d) $y = -4x$

Answers: correct **25** partially correct **48** incorrect **12**

Analyzing of some solutions of students:

Students substituted several values for x and they found values of y as a solution of a given equation. Then they locate the points on a coordinate plane. They had a correct solution. Others only guessed.

8. Draw the graphs of functions

a) $y = x^2 - 4$ b) $y = -x^2 - 1$

c) $y = x^2 + 4x + 4$ d) $y = -x^2$

Answers: correct **34** partially correct **32** incorrect **19**

Analyzing of some solutions of students:

Students substituted several values for x and they found values of y as a solution of a given equation again. Then they locate the points on a coordinate plane. They had a correct solution because these students knew that the graph of the quadratic function is a parabola. Others only guessed again.

9. Convert to the indicated units

a) $324{,}58\ mm = \cdots cm$

b) $98{,}7\ m = \cdots dm$

c) $4\ cm^2 = \cdots mm^2$

d) $2.3657\ dm = \cdots mm$

Answers: correct **13** partially correct **33** incorrect **39**

Analyzing of some solutions of students:

Some students had problems with converting of units.

a) $324{,}58\ mm = 3245{,}8cm$

b) $98{,}7\ m = 9{,}87dm$

c) $4\ cm^2 = 40\ mm^2$

d) $2.3657\ dm = 2365{,}7mm$

The most common mistakes:

- Students have a problem to modify the complex fractions
- They make numerical errors
- Students have problems mathematically express the conditions
- They have problem to solve more difficult examples, which require several mathematics procedures
- Students don´t know read the text carefully
- Students do not know the formula well and often unable promptly select the right one needed to calculate
- They have little developed spatial imagination
- They cann´t draw enough visual image
- Very often they make numerical errors
- They have problems with converting of units
- They do not specify the conditions under which this term has meaning
- They have problems to adjust polynomials (e.g. the removal of braces, as well as setting aside factor-1 before brackets)
- They make wrong steps in operations with fractions

- They make frequently mistakes when they shortcut the fractions. Students without embarrassment shortcut only one addend in the sum (e.g. $\frac{2a^2+13b}{7a^2} = \frac{2+13b}{7}$ etc.)
- They perform the wrong operations with the opposite polynomial
- They very often make the numerical errors in the substitution of a negative number
- They use dificult procedures (e.g. $\frac{2z+1}{z^2-1} + \frac{2+3z}{z-1} = \frac{\dots}{(z^2-1)(z-1)}$)
- Students are not used to verify the result obtained
- The problem is often to write a term $\frac{t^2}{9} - l^2$ as one fraction
- They do not use the correct formula (e.g. $(t-9l)^2 = (t-9l).(t+9l)$ or $(a+b)^2 = a^2 + b^2$)
- Very often are leave out the brackets (for example $x + 1 . x - 1$ instead of $(x + 1) . (x - 1)$)
- The biggest problem for students is the verbal text to express mathematicaly
- Students not thinking about the credibility of the result and they often take for granted completely nonsensical result
- Students often lack a creative approach to solving the problems

Meaning of all these errors is to provide teachers what they should be examine. Identified deficiencies should lead to a search of causes and their gradual removal.

Conclusion

Mathematical knowledge is fundamental to the understanding and development of science and technology as well as being applicable to many areas in the social sciences. We see in the last few years the rapid decreasing of the mathematical knowledge and in general the education level of students in higher education. My experience is the same. It is also alarming that the skill to apply mathematics and the fundamental mathematical knowledge of graduated engineers show a decline [3].

One of the keys to enhanced progress in mathematics for young people is a sound mathematical foundation at primary level, delivered with enthusiasm, confidence and flair. The most important element in mathematics education is teacher [4]. The role of the teacher and school is crucial in people progress as the best teachers are able to motivate, educate and improve understanding for children from a range of backgrounds. The teacher controls the course of teaching and learning, guides and helps the students´ understanding and sets the class goals. The best teachers are able to motivate, educate and improve understanding for children a range of backgrounds. But teachers are often concerned with is student´s success in exams, not student´s understanding of and use of mathematical concepts [1]. If teacher develops student who think mathematically, his class will be really enjoyable for both him and the student. Mathematical thinking is an important goal of schooling. It is important as a way of learning mathematics and for teaching mathematics too. Mathematical thinking is a highly complex activity. Let us strive to teach for understanding of mathematical concepts and procedures, the **"why"** something works, and not only the "how". The relationship between the "how" and the "why" - or between procedures and concepts - is complex. **One doesn't always come totally before the other**, and it also varies from child to child. And, conceptual and procedural understanding actually help each other: conceptual knowledge (understanding the "why") is important for the development of procedural fluency, while fluent procedural knowledge supports the development of further understanding and learning. Teacher should often test a student's understanding of a topic by asking him to produce an example, preferably with a picture or other illustration.

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The Graphic Visualization – the Part of the Solution of Applied Mathematical Tasks

Dana Országhová*

Department of Mathematics, Faculty of Economics and Management, Slovak University of Agriculture in Nitra, Tr. A. Hlinku 2, SK - 949 76 Nitra

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Abstract

Graphical representation of data and processes in various scientific fields and also in different areas of practical life is useful way for their better understanding. In the paper we concentrated on applied tasks focused on economics, especially in the area of detecting properties of functions. The second area is the usage of the definite integral and its application in the calculation of the consumer surplus and the producer surplus. Via solving of applied tasks using a graphic visualization students gain the necessary knowledge and skills for the study of specialized economic courses. We present the results of the brief knowledge survey which was aimed on mentioned mathematical topics. Graphical sketch of the task solution with application, as the output of solving, help students to create and understand the relation between abstract theoretical term and its practical usage. Collected results confirm that graphic visualization provides opportunities for innovation in the content and methods of teaching mathematics.

Keywords: Mathematics education, functions, definite integral, economics application.

Classification: D20, I20, I50, M40

Introduction

At present an important part of education is the use of quantitative methods and tools of information technology in solving economic tasks, the subsequent application in production and trading companies in the analysis and decision making, in banks, insurance companies, the stock markets and in other areas of production and non-production sectors. In the curricula of students of economical faculties are included subjects like mathematics, statistics and computer science, which provide the necessary basis for solving problems with applications.

Information technologies (IT) are changing the way of acquisition, processing and handling of knowledge and also call for changes in the usage of information. This situation is also reflected in the educational process. The IT phenomenon is very important when working with information and educational aspects of this phenomenon were summarized by Hrmo - Krelová [1] as follows:
- Visualization, which facilitates the imagination of the phenomenon and shortens the learning process,
- Process simulation, which may under different input values to create a model of the behavior of the real process,

*Corresponding author: dana.orszaghova@uniag.sk
DOI: 10.17846/AMN.2015.1.1.107-112

- Interaction between the computer and the user, which is one of the important characteristics of multimedia,
- Interpretation of the curriculum, especially in presentations of inaccessible objects and phenomena.

To the mentioned facts we can add that via methods, forms and means of informatics education can be promoted self-discipline of students, their autonomy and creativity. IT tools offer teachers opportunities to motivate students to improve their learning outcomes and promoting study activities generally. If the mathematical concept is explained and also visually represented, then the proper understanding and ability to operate with him in solving practical examples is easier. Therefore, graphs and visualization of solutions in applied problems is an important part of teaching mathematics with application. It is also necessary to provide the study subject with current study literature and use it during seminars [2], [3].

Many studies have dealt with the relationship of students to the tools of information technologies and their ability to use them. The relationship of students to new technologies was investigated by Mišútová - Mišút [4] and their research has confirmed the validity of the hypothesis: there is a positive correlation between attitudes to new technology in the mathematics teaching and performance in tests involving tasks aimed at the basic factors of creativity.

Students of the Slovak University of Agriculture in Nitra come from all kinds of secondary schools. At the Faculty of Economics and Management the majority of students come from grammar schools and business academies. They are expected to study terms and methods of higher mathematics with their applications in the other subjects of specialization. However, levels of their mathematical knowledge from secondary schools are different. In this paper we focused on applied tasks with graphic visualization and students abilities to use them in the solution of mathematical tasks with economic application. We present selected tasks with application in which we can use graphical visualization. The results of realized survey give us information about students' knowledge and skills to use graphs of functions in solving tasks.

Types of applied tasks with graphic visualisation

Economics uses not only basic elementary mathematical operations, but also knowledge of higher mathematics. Because mathematics is not one of the most popular subjects among students and, therefore, teachers strive to make it more attractive for students by means of:
- Implementing of proper, especially applied problems and tasks into the curriculum,
- Proper presentation of the subject via multimedia and Internet which have a global positive impact [5].

Students learn to calculate mathematics in specific mathematical situations tasks purely of mathematical content, which lacks real context. The result is that the student controls theory and can compute tasks [6]. One of the main tasks of education should be the increase in student's motivation to learn. Based on the obtained results it can be concluded that learning objectives are not clear to students, who probably assume that the study subject is not needed for their further education and professional application [7].

The principal topics in compulsory mathematics courses at the Faculty of Economics and Management (FEM) are functions. Plotting graphs of functions belongs to the content of the

first lessons. In the tasks with applications by functions will be represented: costs, revenues, expenses, earnings, consumption and so on. The compulsory mathematical subjects (in the winter and summer semester) include the following topics with graphs of functions:
- An overview of elementary functions - properties and graphs,
- Asymptotes of the graph of a function,
- Detection properties and graphs of functions - monotonicity, concavity.
- Calculation of definite integral,
- Calculation of the area of a plane figure.

Samples of application tasks

1 Functions and applications in economic analysis

Basic functions of economic analysis are often expressed through elementary functions. As an example, we give the total cost function $CN(x)$, which can be formally written in the form of a polynomial of the third degree.

Thus, the expression for a function is:

$$CN(x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3,$$

$$CN(x) = N_f + N_V,$$

$N_V = A_1 x + A_2 x^2 + A_3 x^3$ - is a variable component of the total cost,

$N_f = A_0$ - is fixed component of the total cost.

In Figure 1 and Figure 2 there are graphs of polynomial functions displayed by software GraphSight v.2.0.1. [8].

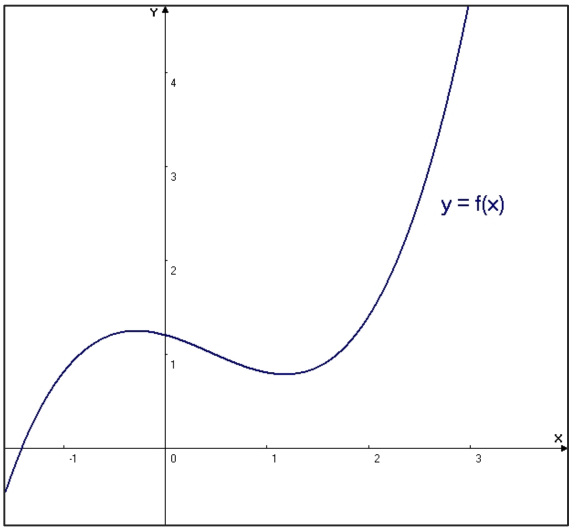

Figure 1*:* $f: y = 0.3x^3 - 0.4x^2 - 0.3x + 1.2$

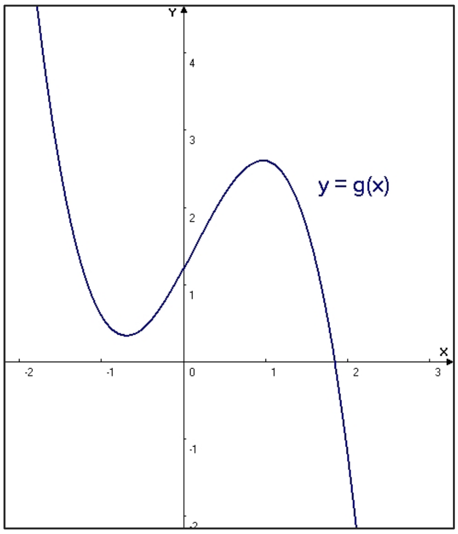

Figure 2*:* $g: y = -x^3 + 0.4x^2 + 2x + 1.2$

2 Definite integral and its applications - the consumer surplus and producer surplus

A consumer surplus (*CS*) is defined as the amount of money that the consumer is willing to pay more for a given product than the current market price. Producer surplus (*PS*) is an economic measure of the difference between the amount that the producer of a given goods receives and the minimal amount that the producer would be willing to accept for the goods. The producer always tries to increase his producer surplus by trying to sell more and more at higher prices.

In the scheme of market mechanism (see Figure 3) there are used these main concepts: $D(x)$ is a function of demand, $S(x)$ is the supply function, the equilibrium point is $E=[\bar{x},\bar{p}]$ where $\bar{x}$ is the equilibrium quantity and $\bar{p}$ is the equilibrium price that satisfy relations $\bar{p}=S(\bar{x})$, $\bar{p}=D(\bar{x})$.

Then formula for consumer surplus is $CS=\int_0^{\bar{x}}[D(x)-\bar{p}]dx$.

Formula for producer surplus is $PS=\int_0^{\bar{x}}[\bar{p}-S(x)]dx$.

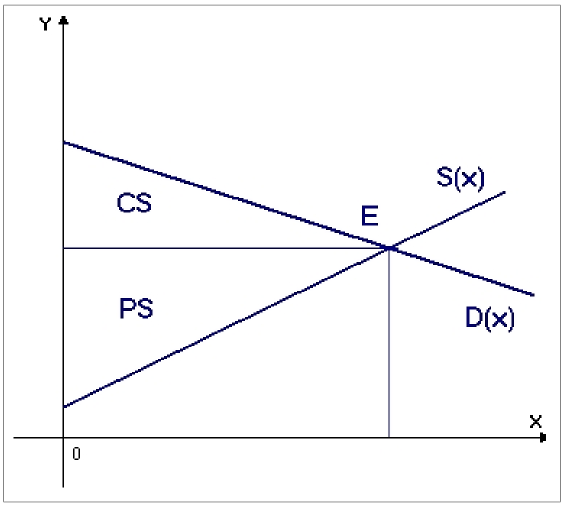

***Figure 3**: Scheme of market mechanism*

Knowledge survey

Higher education of future economists and managers includes the subject Microeconomics, in which the knowledge about the market mechanism is used. We decided to find out the level of the skills and knowledge the first-year students from the main concepts in this area.

During academic year 2014/2015 we performed a survey, which was focused on solving problems with graphical interpretation. The tasks in the test were solved by students of study program Business Economics (number of respondents 60) and students of study program Economics and Management of Agro-food Sector (number of respondents 20), studying in 1st year at SAU Nitra in full-time study. Thus, the number of tested students together was 80. The number of tasks in the test was 4 and scoring of each task was from 0 to 6 points. For visual comparison of the results, we divided the students' score results into intervals < 0, 2.5 >, < 3, 4.5 >, < 5, 6 >.

So we have created a test with the following tasks:
- To find and formulate the properties of the function from its graph (monotonicity, concavity, boundedness),
- To sketch the model of the market mechanism: demand – supply,
- To interpret the found results: equilibrium quantity, equilibrium price,
- To calculate consumer and producer surplus via definite integral.

Results and discussion

Graphical representation of the survey results by the intervals is in the Figure 4. The 1st and the 2nd tasks are related - examine students' knowledge about graphs of functions and their properties. Demand and supply function were given by formula for linear and quadratic function. From the graph in Figure 4 we see that more than half of respondents obtained less than half of the points for these tasks. 3rd task is in relation with 1st and 2nd task, too. The equilibrium point and its coordinates are calculated from the equation $S(x) = D(x)$. The results indicate that more than half of respondents had difficulties in solving this task. We can state that the best results students obtained in 4th task. Students should use the above mentioned formulas for the surpluses and calculate the definite integral. A lot of numerical errors and missing steps in calculating integrals influenced their point score.

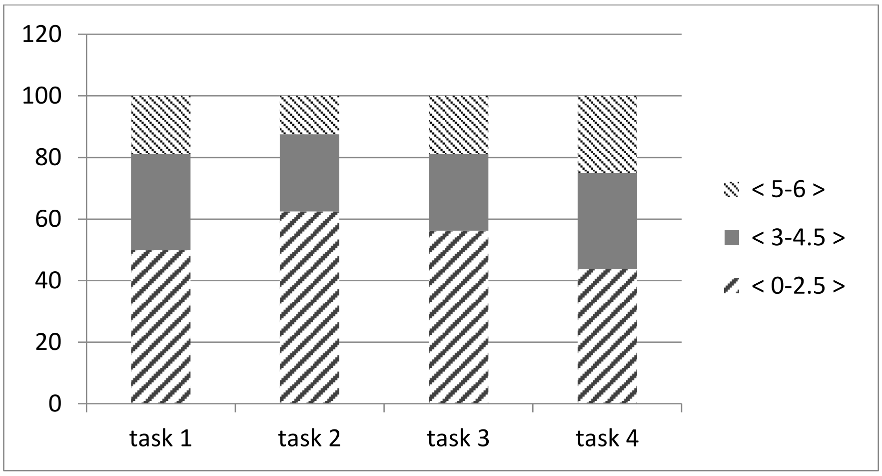

Figure 4: *Results of knowledge survey*

Conclusion

The decision making in the planning, production and sale is conditioned by sufficient information and suitable methods used for solving economic problems. Mathematics provides proper methods and means for solving application tasks and for finding the correct answer. In many tasks students are asked to sketch graph of the function that expresses the dependence between the variables that represent given data. Visualization of the solutions of the tasks, illustrating by graphs, tables and images will have positive impact on the efficiency of mathematical education.

Students perceive the theoretical aspects and practical application of knowledge separately without inner relations. This finding was confirmed by the survey results. Therefore, it is necessary to explain to students mathematical terms not only in verbal form but also through graphic illustrations. Tasks should combine theoretical knowledge with the applications in order to recognize relation between abstract theoretical term and its practical usage.

Graphic illustration and visualization of the solutions provides opportunities for innovation in the content and methods of teaching mathematics:

- The range of fundamental topics and applications should be adapted to the study course,
- To present mathematical methods in applied tasks,
- The solution of the tasks connect with graphic illustrations and images,
- To add graphical interpretation of the solutions in the assignment of seminary projects.

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The Analysis of Students' Solutions of the Word Problem

Lucia Plothová*

Department of Mathematics, Faculty of Natural Sciences, Constantine the Philosopher University in Nitra, Tr. A. Hlinku 1, SK - 949 74 Nitra

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Abstract

The paper deals with the analysis of students' solutions of the word problem with real – life context. The solution of task required creating two mathematical models of real-life situation. The first one was the linear equation and the second one an algebraic expression. Individual solutions created the students of 1st grade of four-year program of grammar school. We use a priori analysis to analyse the students´ solution of our tasks. In doing so, we concentrate on the individual phase of the word problem solution. In this way, we wanted to verify not only the use of all parts of the word problem solution by the mentioned students, but also their ability to create mathematical models of the situation. The paper summarizes the results of a priori analysis exemplified by tables and charts.

Keywords: Word problem, a priori analysis, linear equation, algebraic expression.

Classification: D70

Introduction

According to The Mathematics National Curriculum ISCED 2 (The International Standard Classification of Education), the result of teaching mathematics should be the correct use of mathematical symbols, the ability to read continuous text containing numbers, dependencies and relationships with comprehension. The ability to read incoherent texts containing tables, graphs and diagrams. The teaching of mathematics should lead to building the relationship between mathematics and reality, to gaining experience with mathematization of real-life situation and with creating mathematical models. Some students beginning secondary schools have a problem with the solution of word problems with real-life context. They often lack the ability to model real-life situations and the ability to use the language of symbols.

Word problems

Word problems are irreplaceable in the teaching of mathematics. Word problems are helpful tools to practise the gained knowledge. According to Križalkovič (1968) mathematical word problems are problems which express relationship between given and searched numbers by a word formulation. Hejný (2003) defines a word problem as a mathematical problem which requires language comprehension and overlap with life experience. Odvárko (Novotná, 2000) defines a word problem as a task which includes objects, phenomena and situations (with various characteristics and relationships) from the

*Corresponding author: lucia.plothova@ukf.sk
DOI: 10.17846/AMN.2015.1.1.113-118

different none-mathematics areas. These areas include a common everyday life, various scientific fields and technical practise. Through a word problem students get to a situation where they cannot use any of learnt algorithms immediately. With a task in a word problem students have to find a mathematical definition that fits the given situation the best.

According to Blum and Niss (Šedivý, 2013) word problems should be included into the teaching of mathematics because:

- they are a suitable means for development of general competences of students and attitudes towards mathematics,
- it provides to the students a possibility to be independent in "seeing and judging" and to analyse and understand the use of mathematics,
- it develops both students´ skills in using of mathematical knowledge and also skills in non-mathematical situations,
- it helps students to recognize, understand and keep the mathematical concepts, methods and results.

The word problems solved with the use of linear equations in ISCED 2

Students are meeting the word problems from primary school. At the beginning students solve an easy linear equation with the following methods: trial and error method, table method or intentional object manipulation. Later, in the eighth grade of primary school students meet following concepts: expression, unknown and variable. The tasks are still solved by reflection or illustration. The 9th grade is the grade when the word problems focused on linear equation are solved with the use of equivalent modification.

Methods

We are aiming at the analysis of written answers on word problems of 1st grade grammar school students. At the beginning of the school year the following word problem with real-life context was solved by 23 students:

Water and sewage
Water is the most spread liquid in the world. Before its drinking it is necessary to clean and do disinfection (chlorine). Water supply and draining to the household is provided by *Water Company Clean water* and the company charges the following items for its services:

Table 1*: The price of water and sewage*

Service	Price Euro/m^3	
	Nett price	Price with VAT
water	1,00 €	1,20 €
sewage	0,75 €	0,90 €

Note: Rainfall water fee (the price charged for rainwater drainage from the roofs to public sewerage) is charged according to a roof area in a following way:

- a house with a roof area up to 100 m^2 is charged a flat fee 64,80 Euro per one year
- a house with a roof area over 100 m^2 is charged a flat fee 75,00 Euro per one year

Task 1. Three-room house with a roof area 98 m^2 will consume x m^3 of drinking water per one month. What is a monthly cost of drinking water in three-room house?

Solution: Consumption of drinking water per one month .... x m^3
Water (consumption x m^3 of drinking water per one month) .... $1,2x$ €
Sewage (consumption x m^3 of drinking water per one month) .... $0,9x$ €
Rainfall per one month .... 64,8 : 12 = 5,4 €

$$1,2x + 0,9x + 5,4 = 2,1x + 5,4$$

Monthly cost of drinking water x m^3 is $(2,1x + 5,4)$ €.

Task 2. Mr. Krasňanský pays to the Water Company for his four-bedroom house with a roof area 101 m^2a deposit fee 30 Euro. How many m^3 of drinking water can Mr. Krasňanský consume without any extra pay per six months?

Solution: Consumption of drinking water per one month .... x m^3
Water (consumption x m^3 of drinking water per one month) .... $1,2x$ €
Sewage (consumption x m^3 of drinking water per one month) .... $0,9x$ €
Rainfall per six months .... 75 : 2 = 37,5 €
Accountable advance per six months .... 30 . 6 = 180 €

$$1,2x + 0,9x + 37,5 = 180,$$

from here

$$x \doteq 67,86 \text{ m}^3.$$

Mr. Krasňanský can consume 67,86 m^3 of drinking water without any extra pay per six month.

The main task consists of the text part followed up with two partial tasks. The partial tasks are aimed creating of mathematical model of the situation in the form of an algebraic expression and an easy linear equation.

Before the task was given to students and their solutions were analyzed, the sample solutions of subtasks had been prepared. The a priori analysis of the expected way of solving the subtasks had been done. The variables of a priori analysis of Task 1 and Task 2 were determined (see Table 2 and Table 3). Then, with respect to the variables, the individual student solutions were analyzed.

***Table 2**: The variables of a priori analysis of Task 1.*

	The variable of a priori analysis of Task 1
P.1	The student solved the Task 1
P.2	The student made a full record of task
P.3	The student created a mathematical model
P.4	The student simplified an algebraic expression
P.5	The student wrote an answer to question

***Table 3**: The variables of a priori analysis of Task 2*

	The variable of a priori analysis of Task 2
P.1	The student solved the Task 2
P.2	The student made a full record of task
P.3	The student created a mathematical model
P.4	The student solved a linear equation correctly
P.5	The student verified the accuracy of solution
P.6	The student wrote an answer to question

Discussion

Figure 1 represents a presence of particular variables in students´ solution of Task 1. We marked the variable "Yes, it is correct" in a following case: the statement substituted by the variable was in the solution and the student used it properly. In case the statement substituted by the variable was used in the solution but improperly, the variable was marked as "Yes, but incorrect". A variable that was not in the solution was marked as "No".

In the Figure 1 we can see that 1st task was solved by 21 students. 21 % of them made a full record of a world problem. Other students only read a task and started to do numerical solution of it. In the analysis we can also see that 7 students did a correct mathematical model of a situation.

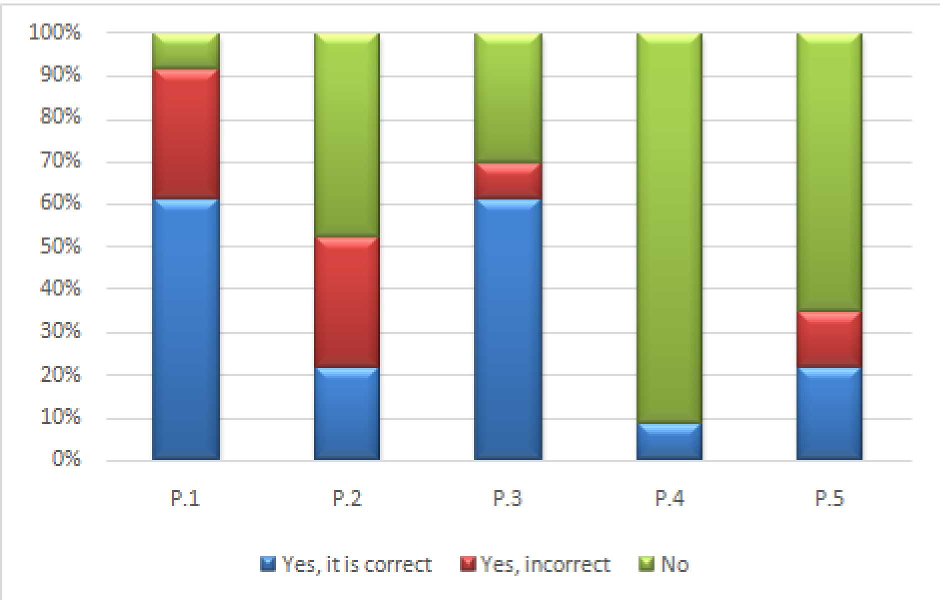

Figure 1

More students were wrong in their precondition that the solution of the task is the price of the rainfall water fee per month. Only two students' solution brought edited algebraic expression representing the price of water and sewage while consumption of potable water is x m^3 per month. The most of students did not write an answer to question because the solution of the task was an algebraic expression.

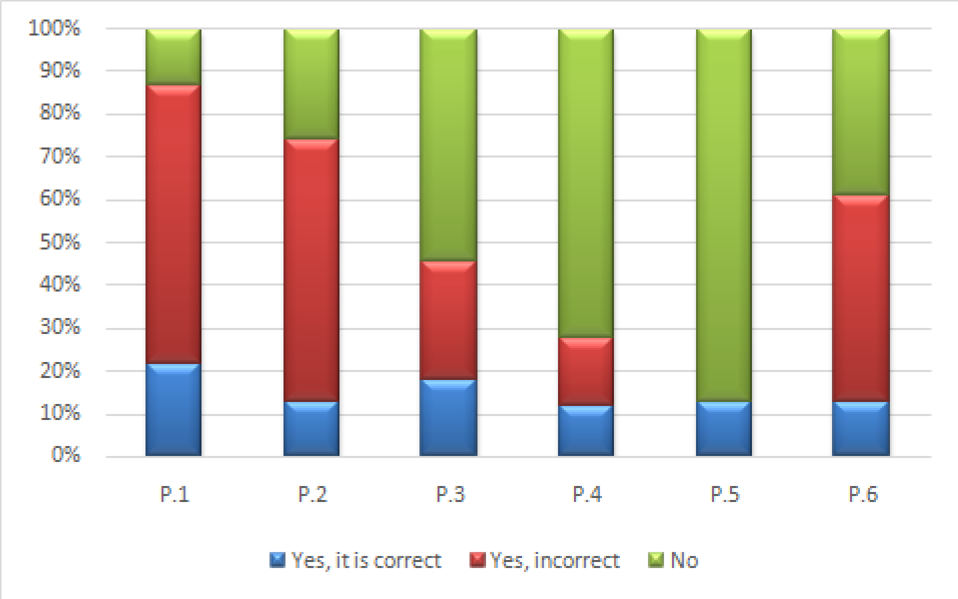

Figure 2

2 students did not solve the Task 2. In 6 of written solutions missed a full record of word problem. Most of students have problem to create a mathematical model of a situation that was an easy linear equation. Wrong mathematical models did not contain rainfall water semi-annual fee, correct sum semi-annual deposit for drinking water. Students, who created correct mathematical model, did not make mistakes in linear algebraic equation solution. 17% of students solved task incorrectly using arithmetical method, 87% of them did not verify their solution and 39% did not write an answer to question.

Conclusion

Based on analysis of students' solutions it can be concluded that students after leaving primary school do not have internalized phases of solving word problem. Many students have problem with creation of mathematical model in the form of linear equation and with algebraic expression. Therefore it is necessary to develop students' skills to create of mathematical models and manipulation with algebraic expression. These skills are necessary for further mathematical learning in higher grades. The reason for the failed task solution could be a lack of understanding assigned tasks inability to obtain necessary information from task or undiscovered appropriate mathematical apparatus.

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The Examples of ICT Usage to Forming the Mathematical Language's Precision

Tadeusz Ratusiński[a*] – Marek Janasz[a]

[a] *Institute of mathematics, Pedagogical University of Cracow, Podchorążych 2, 30-084 Cracow, Poland*

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Abstract

The forming of the mathematical language's precision is important one of the aspects of teaching the mathematic. It appears that this specific activity uses the advantages of modern technology to some extent. One can hardly ever find any references to this particular issue in the literature. We made an attempt to present an idea to use the elements of ICT in process of forming the mathematical language's precision for the deeper understanding of the geometrical notions. To do so we will use the program GeoGebra and the program WinGCLC (an implementation of the GLC language). The wrong usage of the verbs specifying the geometrical constructions may lead to fallacy of the mathematical notions. The properly prepared ICT tool may be helpful to the teacher in taking care of the mathematical language's precision.

Keywords: didactic of mathematics, process of teaching and learning mathematics, math's language, new technology, geometry

Classification: C50, R20, G10

Introduction

There are four fundamental aspects of teaching mathematics: forming the mathematical notions, solving the problems, leading the reasoning and forming the mathematical language [11]. As it seems, the most difficult component of this process is acquiring and developing skill of proper usage of the mathematical language. The culture of language itself is based on the language correctness, which needs the proper choice of words. Then the significant problem turns up - how to verify the accuracy of the language in relation to the mathematical notions? Unfortunately, the most of the easily accessible teaching aids (based on the modern technologies) isn't orientated towards forming the mathematical language's precision directly. But the support provided by the modern technologies may perform a role of secondary importance – may help to form the mathematical language. In this article we present the examples of such ICT usage.

The situation of appearing of the proper connections between the above mentioned aspects in student's mind is very important in terms of the didactic of mathematic. This interdependence is most noticeable between the proper understanding the notion (and its properties) and the language describing this notion communicatively.

*Corresponding author: ratusita@gmail.com
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One should pay attention to the fact that the mathematical object exists in the virtual world of mathematic and the language is necessary to approximate the virtual object by providing the definition of the notion or the sequence of activities leading to the notion. Therefore one deals with the specific instructions written in the natural language and identified with the determined or selected verb.

Let's analyze the aspect of the relations occurring between this components. Below we present the patterns reflecting the relations between the object, the instruction and the distinguished verb.

The aspect relates to the verification of the level of the mastering the material through the following:

the object <-> the instruction <-> the verb

One can analyze the direction of the relation in two ways.

the object <- the instruction <- the verb

In this case one starts with the verbal form and wants to achieve the proper understanding of the object's notion [13] by using the language-the tool helpful to do so. The mistakes in the language's precision may lead to the situation where the object's notion may even not be reflected.

In the second case the direction is contrary to the noted above.

the object -> the instruction -> the verb

A starting point is the existing mathematical object. One wants to achieve the precision of the language by describing this object. Unfortunately, this way has its disadvantages – the description may be wrong, inaccurate or ambiguous.

Finally the verification in both ways leads to feedback and strengthens the connections between the two the most important components of the process of teaching and learning the mathematic.

The aspect in its description needs the metalanguage. In the case of using the DSG (Dynamic Geometry System) tools the interfaces of used applications may play the role of such metalanguage.

The conditions of observations

This paper describes two examples of ICT usage In developing the correctness of mathematical language. Those examples concern the particular group of the students of the intramural complementary mathematics and ICT master's degree course. The observes participated in the observations as they were teachers.

The student had attended two subjects: "ICT" and "ICT in teaching of mathematic" during the first year of the course, the second subject was more orientated towards the IT usage in the process of teaching and learning mathematics. The student had got to know how to use GeoGebra application as well as WinGCLC long before the observations.

Problem

Let's have a closer look at the example – it's an **elementary constructional problem** – to construct a triangle from three given line segments.

This type of constructional problem is rather disliked by most of the students and even by some teachers as it's assumed that it needs a specific approach to an issue. The problem can be seen at various levels of school education. Nowadays the students use the internet to find help for homework and some solutions of the difficult problems. One can often find some post in discussion boards on the internet written by the users looking for the solutions of specific problems, not only constructional.

One of the users tried to help in solving the aforementioned problem and posted the solution as following:

"Take a segment, may it have an a length. Drive the compass into the end of it and sketch a b radius arc. Sketch a c radius arc in the other end of the segment in a similar way. Anywhere the arcs will cross; there will be the third apex of a triangle."[†]

Nobody moderates such posts in respect of the mathematical language's precision or even the correctness of the language itself. The solution quoted above is rather an instance of free-styled narration with many poorly chosen verbs than the correct and linguistically precise solution of the mathematical problem. Sadly one can note the insensitivity for such comments; nobody is offended by such language. The student asking for help is simply weak quasi-solutions like the one quoted above will only deepen his ignorance and distort his understanding of mathematics. On the one hand the teachers cannot moderate the comments on the internet, on the other they can teach the students the correct mathematical language. The helpful aid is provided by GeoGebra.

The observed group of students attended "Elementary Geometry" subject during the first year of the course and considered an issue of solving the similar elementary constructional problems. It seemed that the students, teachers-to-be, would do their best to develop precise and correct language. It wasn't confirmed in the observations. Sadly, the members of the observed group's language was freely, imprecise, clustered with loaned colloquial words and some mathematical object's notions. As it seems, imprecise language is the result of specific kind of ignorance and negligence developed on the former stages of education. The students follow the same patterns as their teachers; the distortion of mathematics increases.

It has to be realized that the teachers-to-be have to take care of the language's correctness to break the circle, but it is hard, long-lasting and arduous work. Hence the idea of nonstandard DSG program - GeoGebra usage.

[†] In Polish, the correct choice of the verb depends on the context of its use and form of expression. Sometimes the wrong choice of synonym makes the statement incorrect. The verb permissible in everyday speech is unacceptable in a formal speech. These nuances cannot be fully cast in the text translated into English, in which such a significant difference does not occur.

One can find many examples related to developing the geometric notions or discovering the properties of mathematical objects and relations between them in the literature (describing the GeoGebra [12, 14], WinGCLC [1] and DGS program [3] usage) along with the descriptions of the problems solved with those tools. As it seems, those depictions leave the linguistic issue out, even on the official GeoGebra's site [17].

This paper analyses the different issue. The language is the key to describe the activities, operations or the mathematical objects. The DGS program is only a helpful tool dictated by the problem's nature. The examples place emphasis on the language's correctness and not on the problem's solution. One can find the literature dealing with the meaning of language in the didactic of mathematics [6, 7, 9], but the papers related to didactic ICT usage in this particular issue are hard to find.

The Example I – GeoGebra

GeoGebra [15] is an application free of charge and it's available on multiple platforms. It's typical for DGS and it's very popular so we can pass its characteristic over. Later we will only be referring to the several basic operations.

Coming back to the initiated problem we should analyze the language of the classic constructions using a pencil, straight-line and a compass. It turns out that even the superficial analyze of the syntax shows the significant association of the verbs and the tools listed above.

In concrete terms with a pencil it's possible to:

- **choose** the points of the object or **name** them
- **find** or **determine** the object's intersection

Adding a ruler (as straightline) it's possible to:

- **lead** or **draw** the line through two existing points
- **connect** two distinct points into the line segment
- **lead** a ray from the one initial point through the second
- **construct** a polygon

Adding the compass it's possible to:

- **draw** the circle through one point with centre another point
- **draw** the circle with the distinct centre and the distinct radius
- **draw** the angle from the distinct point
- **transfer the measure** of the line segment

It appears that the list of possible construction is short. Every one of the procedures requires the distinct verb. One can find all of the mentioned above procedures among the many of the procedures in the GeoGebra application and personalize the options by hiding the unwanted ones and using the ones listed below (Fig. 1.).

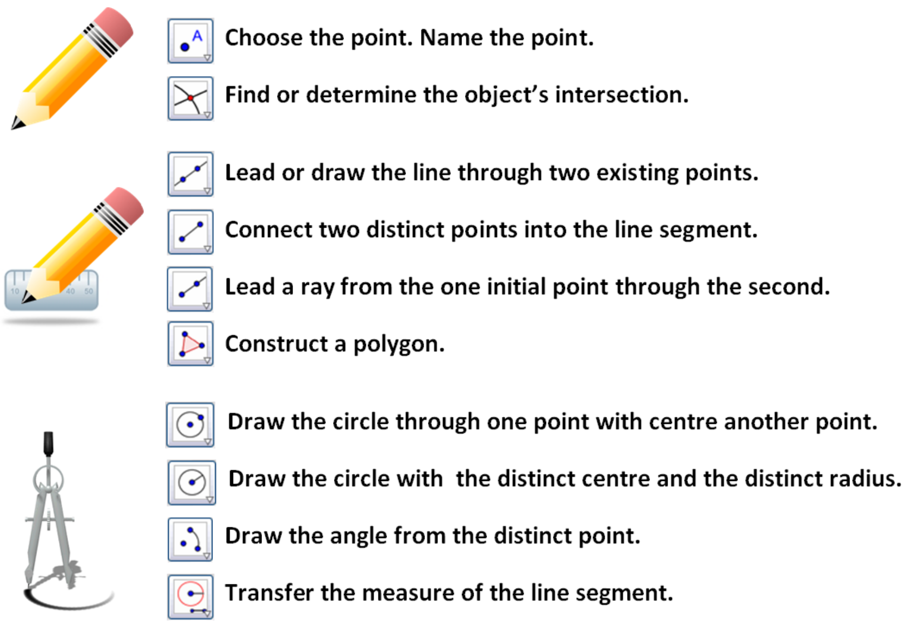

Figure 1

The most important issue of this project is the mathematical object's (shown as "icons" in the program) connection to the proper verbs. This relation in Polish is exact and unambiguous.

This operation leads to forcing the student to use the **specific type of statement during the describing of each the procedure.**

Constructions protocol helps to understand WHAT and IN WHAT ORDER to create to reach the solution. In addiction, helps to understand and remember the type of the mathematical objects developed in the process and to form the culture of mathematic.

Obviously, there are no mathematically incorrect instructions in the GeoGebra application, but we can constantly hear and read such colloquial instructions elsewhere, for example the phrase "to sketch" refers rather to an artistic work than to the mathematical construction. The discussion of the questions –*how to solve the problem in GeoGebra?* or *what kind of icons to use?*-could be helpful to make the students realize the problem and to reveal the mathematical language's incorrectness. It generates the care of the language's precision and the statement's aesthetic. The GeoGebra application clearly states that the line, the ray and the line segment are the very different objects and helps the students to proper classify this objects.

Observations

The observations of the students clearly shown that the exact understanding of the connection:

the object -> the instruction -> the verb

leads to better understanding the notion's properties, to restructure the mathematics, to knowledge's infiltration to different fields of mathematics and beyond. The observed students admitted that it helped them to organize their knowledge.

Another problem is the fact that the complete solution of the constructional problem demands not only the description of the construction itself, but the students often prefer to identify this terms, but this article doesn't analyze that important issue, it needs to be researched and described in future studies.

The Example II – Geometry Constructions Language

Another tool helpful in forming the mathematical language is WinGCLC program-the implementation of Geometry Constructions Language (GCL) [4] on the Windows operational systems with the graphic interface [5]. This specific language was created for constructions of the geometrical objects and for the possibility of describing those constructions on the Euclidean plane. The language has simple and intuitive syntax [10]. The available geometrical constructions are performed with use of the implemented and classic tools-a ruler and a compass.

There are some specific groups of instructions in the language's syntax. In this article we will use only four of them: basic definitions, basic constructions, labeling and printing commands and drawing commands.

The application forces the user to apply the proper syntax of the command, since the user-friendly graphic interface is hard to find in this tool. If the command is incorrect the application will signalize the error; it's the result verifying the correctness. There are three types of such announcements [1]-syntactic errors, semantic errors and deductive errors. The user is determined to apply the proper chosen commands connected with the definition of the constructed object. There is no ease in choosing the keywords; the keywords are the commands-the verbs. It determines the formal notation, but no instrumentally.

The below given example illustrates the notation of the program in the GCL language describing the solution of an elementary constructional problem - let's use the problem quoted before: to construct a triangle from three given line segments (Fig. 2.).

point A 50 40	**drawcircle c1**
point B 80 40	**drawcircle c2**
point D 80 40	**intersec2 C1 C2 c1 c2**
point E 100 40	**cmark_b C1**
cmark_l A	**cmark_t C2**
cmark_r B	**drawsegment A C2**
drawsegment A B	**drawsegment B C1**
circle c1 A D	**drawsegment A C1**
circle c2 B E	**drawsegment B C2**

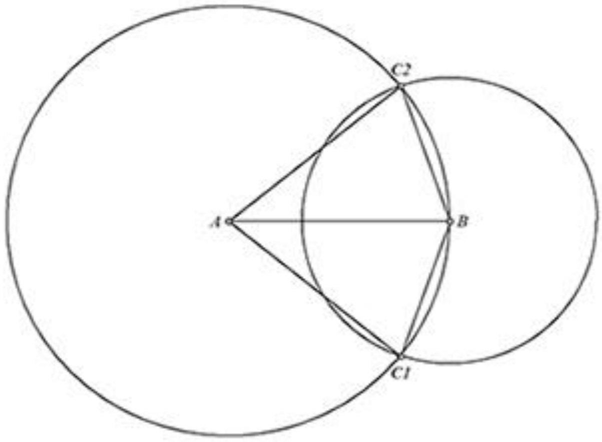

Figure 2

The language's syntax is simple and intuitive enough to assume that the analyze of the aforementioned program won't make an issue and it will be easy to read the idea of connection between the command and the defined object. Therefore we will leave the wider discussion out and refer the curious readers to the program's documentation [16].

It the situation above the verification of the absorbed knowledge leads from the object to the verb. The example will be provided by the following case-on the one of the stages of the construction two distinct auxiliary circles don't have an intersection; the program generates the error result, which is suggesting its essence and the cause in relate to the mathematical object instead of the language's syntax.

Observations

In this program the word means the object's construction. The aspect

the object <- the instruction <- the verb

is highlighted then. In case the program notices some errors or distortions the object isn't responding or it does not exist. The language correctness verification needs the ability of "reading the object".

The students emphasized repeatedly that it seemed WinGCLC language needs the programming abilities rather than mathematical ones, but it appeared that this process is only reverse to the one from the first example. This one demands describing of the constructions protocol with respect of every assumptions and relations between the objects. To construct and to manipulate the protocols one needs to develop the deep understanding of the mathematical objects AND the ability of precise description of these objects. The feedback from the program contains suggestions related to statement's verification and controls the errors. But the proper knowledge is the most important to correct language incorrectness.

Conclusion

The aforementioned examples illustrate the aspect of the verification the absorbed knowledge, but the courses of the examples are opposite.

In the first one (GeoGebra) one can observe the programming of the verification from the distinct verbs to the ideal objects, care of the language's correctness influences the process of forming the object' notion in student's mind very positively.

In the second one (GCL) the verification runs from the constructed object to the description. The errors signalized by the GCL language's processor force the correctness and the precision of the language. But it is noticeable that the correctness is formal in the GCL language made to describe the geometric notion and its properties. Hence it should be stated that the DGS is not only the operation on the program's graphic interface, but it's also an environment using the visual language.

The wrong usage of this tool (without the care of the language's aspect) may lead to an unwelcomed results, like the forming the distorted image of the object or gaining to habit of the distorted formalism.

Not only the solution of the problem, but the annotations also are desired on the every educational stage. The relation between the mathematical objects and the language is natural and strict. The examples of use the ICT to forming the mathematical language's precision show that this relation is not only formed , but also verified.

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Developing and Piloting an Instrument for Measuring Upper Secondary Mathematics Teachers Beliefs in Nitra Region

Ján Šunderlík[a*] – Ľubomír Rybanský[a]

[a] *Department of mathematics, Faculty of Natural Sciences, Constantine the Philosopher University in Nitra, Tr. A. Hlinku 1, SK - 949 74 Nitra*

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Abstract

In our contribution we present the design and piloting process of instrument for measuring mathematics teachers' educational beliefs in Slovak educational settings. We briefly describe the theoretical framework on which the presented questionnaire is based. We also present the preliminary results of data collected from upper secondary mathematics teachers in Nitra region and the process of factor developing. For data analysis and reduction of items we used principal component analysis. Than we correlate the obtained factors to see the modules consistency. We present the particular parts of the questionnaire and some preliminary results of three selected modules of the questionnaire.

Keywords: Teachers' beliefs, PCA, in-service teachers.

Classification: C19,C29

Introduction

Investigation of mathematics teachers' beliefs is one of the international research interests in mathematics education. This focus is mostly influenced by the mathematics educational reform that oriented mathematics education more to student-centered education and problem solving on national level. Within these settings there have been new demands on mathematics teacher as one of the key factors of successful educational reform. Important, mostly labeled as "hidden", variable in this educational transformation are beliefs, because they reflect in what way mathematics and its teaching and learning is conceptualised by teachers. Thompson states that „what a teacher considers to be desirable goals of the mathematics program, his or her own role in teaching, the students' role, appropriate classroom activities, desirable instructional approaches and emphases, legitimate mathematical procedures, and acceptable outcomes of instruction are all part of the teacher's conceptions of mathematics teaching" (Thompson, 1992, p.135 in Lepik, Pipere 2012).

In comparison with the international research there is still lack of empirical data mapping and analyzing mathematics teachers' beliefs structure within the Slovak educational settings. In our contribution we aim to describe the theoretical background and pilot the instrument in Slovak educational settings. The instrument was prepared in international collaboration

[*]Corresponding author: jsunderlik@ukf.sk
DOI: 10.17846/AMN.2015.1.1.128-134

for measuring of aspects of upper secondary mathematics teachers' beliefs concerning job satisfaction, teaching, school mathematics and mathematics didactics.

Theoretical framework

In our contribution we understand teachers' beliefs broadly as "conceptions, view and personal ideologies that shape teaching practice" (Lepik, Pipere & Hannula 2012). It is assumed that what one beliefs to be right influences what one does – beliefs act as the teacher's pedagogical predispositions. Thus, we consider beliefs as factors shaping the teacher's decisions, for example, about what goals should be accomplished and how should the effective learning of mathematics look like (Schoenfeld, 1998 in Hannula & al. 2013). We understand beliefs as regulating system closely connected with teachers everyday practice that influences instant decisions they need to make every lesson. Based on the research, we can assume that there are several time periods when and how the mathematics teachers' beliefs are developed. There is a suggestion that teachers start their careers with previously constructed and possibly subconscious theories about teaching (Powell, 1992 in Lepik, Pipere & Hannula 2012). The important time when beliefs development may be influenced is the period during the teacher preparation. As (Šunderlík, 2010) suggested, during teacher preparation also the teacher's identity is developed. Within this process the implicitly held beliefs, from own school years at secondary school, may be questioned during the teacher preparation, especially during their student teaching. Supporting pre-service teachers' reflection skills enables prospective teachers to enact their own implicitly held beliefs in their teaching. Furthermore, as Clarke (1988 in Lepik, Pipere & Hannula 2012) suggests, teachers continue to hold their implicit theories throughout their careers. In our study we are interested in mapping this stable experienced mathematics teachers' beliefs system. Considering the inner character of the beliefs system, we understand that it is hard to make a direct insight into this system. That is why we narrow our target to the teachers' openly acknowledged explicit or espoused beliefs (what is said) designating what teachers think about the impact of teaching in general, as well as their understanding of how children learn, being aware of the potential inconsistency between the espoused beliefs, less conscious implicit beliefs, and beliefs in action or enacted beliefs demonstrated in the consistent behavior (McMullen in Lepik, Pipere & Hannula, 2012).

Based on the research, we measure the level of two major constructs of general and mathematics teachers' beliefs about the nature of teaching and learning. Within the general beliefs about good teaching and learning we include "direct transmission beliefs about learning and instruction" or so called, "traditional beliefs" and "constructivist believes about learning and instruction" (OECD, 2009). However, recently some voices have appeared, challenging any dichotomisation in educational studies, and especially in international comparative research, , suggesting the complementary view on the interrelated nature on teacher-centred vs student-centred classrooms. (Andrews & Sayers, 2013; Clarke, 2006 in Lepik, Pipere & Hannula, 2012). Within the mathematics teachers' beliefs we consider three mathematical constructs that are described in Rösken & Liljedahl (2006). These constructs can be briefly characterized as follows: "Pupils should have an opportunity to independently develop their mathematical understanding and knowledge" (Process), "In a mathematics lesson, there should be more emphasis on the practicing phase than on the introductory and explanatory phase" (Toolbox); "Working with exact proof forms is an essential objective of mathematics teaching" (Proofs).

Research focus

In our contribution we pilot the instrument for measuring mathematical teacher beliefs about good effective teaching of upper secondary mathematics teachers. For this purpose it is necessary to verify how the theoretical construct of general teacher beliefs and mathematics beliefs about good teaching works within the Slovak educational settings.

Methodology

For data collection we used questionnaire that was developed to be valid in cross-cultural way to measure different aspects of teachers' mathematics-related beliefs. The questionnaire was based on the NorBa project questionnaire prepared for lower secondary mathematics teachers (Lepik & Pipere, 2012). This tool was adapted for upper secondary mathematics teachers. Some questions were deleted, new items about using ICT and new module F focused on views about mathematics was added. The modules of the final version of the questionnaire describe A) general information; B) teachers' overall job satisfaction; C) views of two teaching approaches; D) views about good teaching; E) teachers' conceptions of good teaching/learning of mathematics; F) views about mathematics; G) and questions about typical classroom practices. Different modules give us a unique opportunity to set up separate factors for each area of interest, and then compare them within the modules to the obtained complex view about teachers' mathematical beliefs structure. In our contribution we focus on analysis of modules C, D and E.

Module C consists of two descriptions of mathematics teaching that is focused on teaching combinatorics. The first teacher represents mostly the traditional way of teaching, whereas the second teacher represents mostly the constructivist way of teaching. After these two situations, there are four questions asking teachers which of the two teachers they prefer. Module D is focused on teachers' general beliefs on teaching and learning. It consists of 23 Likert-type items. The module E measures teachers' beliefs on mathematics teaching and learning. It contains 20 Likert items from (Pehkonen and Lepmann, 1994 in Hannula, Lepik, Pipere, Tuohilampi, 2013). The items are focused on three different constructs (see above): system, toolbox and process. The questionnaire was prepared in English and then translated into Slovak. The translation was performed simultaneously by two translators and checked by a math educator fluent in English and an external translator who also did the qualified Slovak proof-reading to ensure the best possible translation of all items.

Participants

Data were collected from the upper secondary mathematics teachers in Nitra region (n = 56). The age of this teachers ranged from 31 to 65 (average = 50); the length of service of these teachers ranged from 7 to 44 (average = 26). Within this group there were 33.9% teachers from grammar schools and 66.1% teachers from vocational schools. Their average age was 50 years; 25% of them were male; 35.7% of them were qualified Doctors of Pedagogy; all of them were qualified mathematics teachers; and 34% of them had attended less than 5 days of Professional development. The average number of students in a classroom was 25.

The data collection was completed in February, 2015, in Nitra region. Data collection was realized in cooperation with the regional school authority. Data were collected via online questionnaire that was sent to official school address and was forwarded to mathematics teachers. Collected sample is about 34% of the entire population of upper secondary mathematics teachers. We can assume that our sample is representative because if we compare our sample with the entire population of upper secondary teachers in Nitra region based on survey of Ministry of Education in school year 2014/2015 the median of teaching experience in our sample is 27 years, median of teaching practice of all teachers is 22 years, percentage of women in our sample are 74% within all teachers there are 77% of women and in our sample 33% of teachers teach at grammar school and in entire population there are 37% of teachers teaching at grammar school.

Analysis

In order to reduce data into fewer, but more reliable variables, we used principle component analysis. Modules D and E were analyzed separately with the use of Varimax rotation. The common statistical criteria for PCA were tested that can be find in (Leech, Barret, & Morgan, 2008). Several variables in modul D had to be removed due to low communality or multiple loadings. For clearer picture several solutions with different numbers of factors were tested. The criteria to select the factors were reliability and easy interpretation of the factor.

Results

Upper secondary mathematics teachers' background.

Module C

In module C teachers are asked to read about two approaches of teaching combinatorics. Teacher A used mostly transmisive way of teaching and teacher B used constructivist way of teaching. After that teachers are asked to answer four questions presented in Table 1.

Table 1*:*

Questions related to specific learning goals	**Average response (mean±STD)**	**Definitely A Tend toward A (%)**	**Cannot decide (%)**	**Definitely B Tend toward B (%)**
C1 Which type of class discussion you would be more comfortable having in class?	3.87 (1.11)	20%	7%	73%
C2 Which type of discussion do you think most students prefer to have?	3.58 (1.18)	24%	16%	60%
C3 From which type of class discussion do you think students gain more knowledge?	4.09 (1.06)	15%	2%	83%
C4 From which type of discussion do you think students gain more useful skills?	4.16 (0.98)	11%	7%	82%

In the first question we asked directly about the teaching style teachers prefer; in questions C2 – C4 we asked indirectly about teachers' belifs asking about what they thinks about the influence of two approaches of instruction to students learning. As evidenced by the results of module C, upper secondary mathematics teachers in Nitra region prefer teaching style of teacher B, who represents the more constructivist way of teaching. This pattern is slightly shifted only in question 2 asking for type of discussion students prefer. This is something interesting because constructivist instructions are based on more open student-centered questioning. This inconsistency in pattern is open to several possible explanations. It may be culturally based that students are not used to discussion about mathematics and teachers consider it as if students preferred direct "traditional" question with clear answers. We can suggest that teachers tend to simplify the questions until they get the desired answer. It may be also the case of how long the teachers are willing to wait for a discussion or a complex answer. To answer this question a more accurate and detailed qualitative research need to be conducted. Based on module C results, we assume that about ¾ of upper secondary mathematics teachers prefer constructivist way of mathematics instruction. Quite high percentage can be expected also because reform national curriculum preferer more constructivist way of instructions.

Module D

In module D the 23 questionnaire items were subject to the principal component analysis (PCA) with Varimax rotation. The number of extracted factors was determined by using classical eigenvalues and scree diagrams. Based on these criteria, it was decided to explore solutions of four, three and two factors. The best solution with most obviously interpretable factors was found in two-component structure. This corresponds with (Lepik, Pipere & Hannula, 2012) and also with the above mentioned theoretical framework that reduces the questions in module D to two major factors. Because of low percentage of explained total variance we reduced the number of items and used the same model with thirteen items. The removed questions will be analyzed and reformulated. This two-component solution explained a total of 41.2 % of the variance. The first factor F_D1 was labelled Reasoning and conceptual understanding we labeled it as (constructivist) ($\alpha = .765$). Eight items forming this factor represent a perspective on (mathematics) teaching which emphasizes the students' active and meaningful participation in learning process: students' discoveries and inquiry on problems, working in small groups; aiming at conceptual understanding. The second factor F_D2 was labelled Mastery of skills and facts (traditional) ($\alpha = 0{,}655$). The five items of this factor emphasize (mathematics) teaching as concerned with the formal teaching of skills and fluency through practice of routine procedures; repeating the basic skills; there should be exact instructions to students; during teaching there should be silence in the classroom and foremost the direct transmission of knowledge from the teacher to the pupil. The reliability level of this factor is not so high. This may be because of lower number of items in the factor. In general, this construct remained stable throughout all tested factor-models. Although the reliability of this factor was not very high, we believe that it is a well-defined construct and it can be used for reducing data complexity. Both factors F_D1 and F_D2 seem to be independent components and not opposite extremes of one scale. So, in case of an individual teacher they both may exist in parallel. For example, a teacher who emphasizes discoveries and inquiry on problems in their teaching may also highly value practicing of routine procedures.

Module E

In module E twenty items were subject to the PCA with Varimax rotation. The analysis was performed in the similar way as for module D. The number of extracted factors was determined by eigenvalues and scree diagrams. Based on these criteria, it was decided to explore solutions of four, three and two factors. The best solution was with three factors. In comparison with theoretical framework (Rösken & Liljedahl 2006). There is some overlap between factor system and toolbox within the Slovak upper secondary mathematics teachers. There were also one inconsistency between theoretical and empirical model of factors. Two items that were design to contribute to toolbox factor contribute to system factor. Althoug this small inconsistency we named the factors as system factor. If we look closer to few more toolbox and system items they contribute also to both factors. This results give us reason for later investigation of the relationship between this two factors between Slovak upper secondary teachers in larger sample. Based on the data we named these mathematics teaching beliefs: F_E1 System (α=0,817; 9 items), F_E2 Process (α=0,728; 6 items), F_E3 Toolbox (α=0,692; 5 items). The empirical factors were almost identical with the theoretical factors. The toolbox factor has again lower reliability level that may be caused by the lower number of items within this factor.

Comparison of factors construct

For each teacher we calculate the values of each factor in module D and E. Than we correlate this values to see the relationships between the calculated factors. Based on the three modules we would like to see whether the teacher answers correlate within the modules. We are especially interested in correlation between module D, E and question C2.

***Table 2**: Pearson correlations between view of two teaching approaches, general teaching beliefs, and mathematics teaching beliefs.*

	F_D2	F_E1	F_E2	F_E3	C1	C2	C3	C4
F_D1 (constructivist)	0.00	0.12	0.49**	0.26	0.23	0.29*	0.11	0.08
F_D2 (traditional)		0.34*	-0.25	0.48**	-0.27	-0.40**	-0.46**	-0.42**
F_E1 (system)			0.00	0.00	0.01	-0.09	-0.23	-0.24
F_E2 (process)				0.00	0.26	0.24	0.33*	0.26
F_E3 (toolbox)					-0.07	-0.21	-0.19	-0.19
C1						0.48**	0.74**	0.67**
C2							0.49**	0.40**
C3								0.85**
*. Correlation is significant at the 0,05 level (2-tailed).								
**. Correlation is significant at the 0,01 level (2-tailed).								

In the correlation table we can see that constructivitist way of instruction and positive beliefs towards process oriented teaching mathematics were significantly correlated, r = 0.49, p < 0.01. Next, we can see that traditional way of instruction and positive beliefs tovards toolbox oriented teaching mathematics were significantly correlated, r = 0.48, p<0.01 and also towards system oriented teaching r = 0.34, p < 0.05. We can also see that the factor F_D2 correlates negatively with questions C2 r = -0.4, p < 0.01; C3 r = -0.46, p < 0.01 and C4 r = -0.42 p < 0.01. There is also a small correlation between F_E2 and question C3 and F_D1 and question C2 on α=0.05. In general, these preliminary results show us consistency of similar factors construct within the three presented modules.

Discusion and Conclusion

In our contribution we present the preliminary analysis of three modules of designed international instrument for measuring mathematics teachers beliefs in Slovak educational settings. Based od the PCA analysis we verify the theoretical constructs with empirical results of our data. Also the reliability of presented items within the modules gives us a positive signal of appropriate usage of designed istrument for measuring of espoused beliefs of upper secondary teachers. Based on the presented results we consider this questionnaire to be valid and suitable for further analysis and measuring of mathematics teacher beliefs. The pilot also has shown problematic formulation of several items that will be further analyzed and reformulated.

On the other hand, we need to be aware of inner aspects of beliefs, and that is why for more complex picture it would be beneficial to support the obtained data with observation of everyday practice of mathematics teachers where also contextual factors, teacher knowledge transferred to teachers' competence and school culture can be considered.

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Mathematical Literacy Assesses Student Knowledge

Soňa Švecová[a*] – Janka Drábeková[a] – Lucia Rumanová[b]

[a] *Department of Mathematics, Faculty of Economics and Management, Slovak University of Agriculture in Nitra, Tr. Andreja Hlinku 2, SK - 949 76 Nitra*

[b] *Department of mathematics, Faculty of Natural Sciences, Constantine the Philosopher University in Nitra, Tr. A. Hlinku 1, SK - 949 74 Nitra*

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Abstract

In this article, we highlight the importance of the task creation of mathematical literacy and show that it is not always a simple process. We present that the creation of tasks from mathematical literacy is an evaluation of mathematical competencies, which the teacher knows and later on while solving problems the student can effectively use this knowledge in problematic situations. Mathematical literacy assesses students' abilities to make connections between different topics in mathematics and to integrate real information from everyday life. It's never too late to build math skills and helps students to decipher what they definitely need to understandings of mathematics in their everyday life.

Keywords: mathematical literacy, tasks for student, knowledge.

Classification: D84, M24, M65

Introduction

What does mean mathematical literacy? The process of developing literacy skills is fairly well- known during the childhood. First we learn sounds, then letters, writing, reading and at the end we have a sense create our language. During our life are very important literacy skills, especially mathematical because it helps us to solve real world problems. For the purposes of PISA 2015, mathematical literacy is defined as follows:

„Mathematical literacy is an individual's capacity to formulate, employ, and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts and tools to describe, explain and predict phenomena. It assists individuals to recognize the role that mathematics plays in the world and to make the well-founded judgments and decisions needed by constructive, engaged and reflective citizens"(1).

Finally, modeling tasks are social, reflecting real-world practices, whereby problem solving takes place via a process of interpreting, discussing, explaining, analyzing, justifying, revising, and refining ideas (2).

How to create tasks of mathematical literacy?

To create typical mathematical tasks is not a difficult process for a teacher with experiences where he/she can practice the concrete rules learned by students. While talking about

*Corresponding author: sona.svecova@uniag.sk
DOI: 10.17846/AMN.2015.1.1.135-138

exercises from mathematical literacy, students do not have the connection of relationships, which they need to solve, they have to find them out in the context of each particular task, have to know how to read with comprehension and link the problem with mathematical relationships. Often there is information that is not relevant with the problem and the student should be able to analyze them. Tasks from mathematical literacy should give a recipe how to react in real life situations so the student will explore the beauty of math. Each person can create tasks from mathematical literacy in terms of mathematical competencies. We can say that the creation oftasks from mathematical literacy is an evaluation of mathematical competencies, which the teacher knows and later on while solving problems the student can effectively use this knowledge in problematic situations. Mathematical literacy, it is a sum of mathematical competencies, knowledge of math, which an individual is able to use in various situations. Context of these tasks is natural for the usage of math, influence the problem solving and its interpretation in a difference when comparison with tasks which occur mostly in math textbooks, which main aim is to practice mathematical skills. We would like to show how to create or redefine typical mathematical tasks into tasks of mathematical literacy.

Task 1 is from Thematic content: Terms, functions, tables, diagrams, Thematic unit: Solving equations, systems-linear (3). The student's goal is to solve word tasks, which require solving easy equations with an occurrence of one unknown unit or a system of equations with two unknown units, which could be converted to the one equation to model real life problems with usage of mathematical language and interpret results of mathematical problem into real situations.

Task 1.1

Typical mathematics task: Solve the scheme!

$$10x + 7y = 425$$

$$15x + 14y = 783$$

Solution: $x = 13{,}4 \; and \; y = 41{,}6$

Task 1.2

Exhibition of typical mathematical task as context task:

Text:

The water is flowing to the reservoir with two taps. With the first one, it will flow 80 liters in hour, with the other one only 56 liters per hour. After half an hour of impregnating, the water in reservoir had a temperature of 25 degrees Celsius. If we impregnate water from one tap 45 minutes and from other tap 1 hour, the impregnated water will have temperature of 27 degrees Celsius. Determine the temperature of water of every tap (We assume that the temperature of water from taps does not change while the whole time of impregnation).

Solution: $x = 13{,}4 \; and \; y = 41{,}6$

Task 1.3

Exhibition of typical mathematical task as the task from mathematical literacy:

Text:
Mr. John bought a 165 liters bathtub volume. He fulfilled the bath in two-thirds of the volume with a bathing temperature of 40 °C. How many liters of hot water does he need? How many euro's he pay for whole bath? Use table 1.

***Table 1**:*

Type of water	Price of water 1 m^3 in €	Temperature of water in °C
Cold water	2,56	12
Hot water	12,11	60

Solution: He needs 64,2 liters of hot water. He pay 0,895 €.

Task 2 is from Thematic content: Combinatory, probability, statistics, Thematic unit: Combinatory. (3) The students have a goal to use different strategies of finding out the possibilities by inscribing or systematic inscribing of possibilities or combinatory rule of sum and multiplication.

Task 2.1

Typical mathematics task:

Text:
How many six digit numbers can we create from digits 1,2,3,4,5,7,8 in a way that digits 2 and 3 cannot stand next to each other?
Solution: 480

Task 2.2

Exhibition of typical mathematical task as context task:

Text:
Guitar strings are marked as ***E, a, d, g, h, e***. In which different ways is possible to order these strings if strings ***E*** and ***e*** cannot stand next to each other.
Solution: 480

Task 2.3

Exhibition of typical mathematical task as task from mathematical literacy: Text:

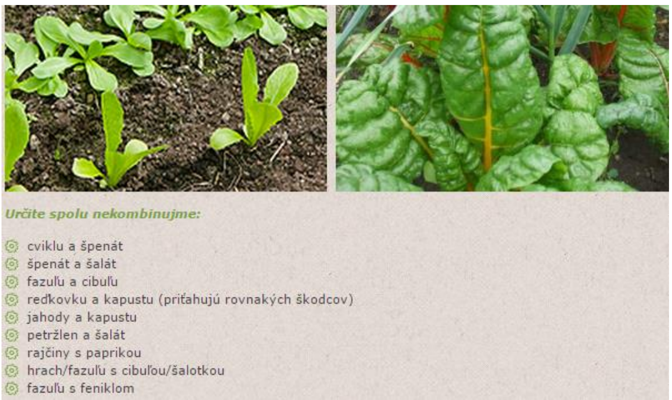

Figure 1: *Vegetables in the garden (http://www.allworks.sk/tipy-a-triky/ako-skombinovat-rastliny-v-zahrade-pri-sadeni)*

Mr. John has decided to plant his garden and wants to plant a beet, tomato, pepper, been and onion there. On the internet, he found information, which plants cannot be combined together. How many ways of ordering his plant in garden does he have if he will all plant them next to each other to rectangle ground plan and he wants to start for sure with an onion?

on				

Solution: 16

Conclusion

In the task, 1.1 and 2.1 students are required to solve the problem by using their memory without logical instructions. Students do not have a clue of why this memorized method is important for their life. Tasks 1.2 and 2.2 give students the opportunity to state the relational pattern in words or mathematical symbols. Questions require students to solve the question that leads to a correct answer. Last tasks 1.3 and 2.3 use mathematical literacy. Students need enough time to understand and analyze the question to be able to choose the right strategy for solving the particular problem. They have to go deep inside into their mathematical knowledge. Mathematical literacy assesses students' abilities to make connections between different topics in mathematics and to integrate information in order to solve simple problems and to make connections among the different representations. In these mathematical literacy tasks, students analyze, develop and interpret their own models and strategies and they are able to make mathematical arguments, including generalizations.

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Geometric Figures in Concept Maps in Primary Education

Valéria Švecová[a*] – Gabriela Pavlovičová[a]

[a] *Department of mathematics, Faculty of Natural Sciences, Constantine the Philosopher University in Nitra, Tr. A. Hlinku 1, SK - 949 74 Nitra*

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Abstract

The aim of the school learning is to acquire and remember the most knowledge and information possible and thus to enlarge own notion system. The process of remembering and acquiring could be facilitated by rewriting key words from the text to a larger piece of paper and connecting the key words into a space structure according to their relation and coherence. In such a way the concept map will arise and may serve a suitable device. Applying the concept mapping as well as other metacognitive methods into the school program would be of double important role. Firstly, such methods may help the future teachers to bring their own teaching approaches nearer to a more logical and senceful way of teaching. They will look for how to elaborate the teaching material and how to make it more clear and understandable. That means they will emphasize the role of key words and key principles as well as their mutual relations and coherences. Secondly, they will do it in a way which will form the learner's contact understanding of the given theme. In this article we were interested in concept mapping at the primary level of education.

Keywords: Primary school, concept maps, square, rectangle.

Classification: C30

Introduction

A teaching process at the primary school level should be a well thought-out, detailed and logical unit in which we achieve certain educational goals. Pupils examine geometric figures as units and they differentiate between them according to their shapes. Then comes an analysis of these geometric figures and the result of this analysis is to set apart their characteristics according to which we differentiate and describe these figures. Based on these characteristics we create correct images and concepts (Šedivý, Križalkovič, 1990).

Concept maps at the primary level of education

In our research, we focused on creating concept maps on the subject SQUARE / RECTANGLE. The concept maps were creating by 4th grade primary school pupils.

Regarding to guidelines ISCED 1 a pupil has to have following knowledge and skills:

- square and rectangle draw in a square grid;
- term and mark vertices of square and rectangle;
- identify properties of square and rectangle and characterize them.

*Corresponding author: vsvecova@ukf.sk
DOI: 10.17846/AMN.2015.1.1.139-144

According to Petrasová (2008) a concept map is a graphic presentation of a pupil´s knowledge structure from a particular subject where knots = points represent concepts and flow lines = arcs/lines stand for relations between the concepts. When designing a concept map a certain scheme – diagram is created. It is a method of studying, finding and testing the knowledge of individual pupils.

Tony Buzan (2003,p. 4) defines concept (thought) maps as following:

1. „A simple way of getting the information into the brain and out again."
2. „A new way of studying and revising which is fast and works well."
3. „A way of making notes that is not boring."
4. „The best way to come up with new ideas and to plan projects."

A concept (thought) map is comprised of words, colours, lines and pictures. It can be created very easily and it can help us:

- remember better and make clearer notes;
- get better marks;
- come up with ingenious ideas;
- save time and use it for a different purpose;
- sort out our thoughts, interests and the whole life (Buzan, 2003)

If a teacher gives his students the task of creating a concept map based on the currently taught topics, he can easily and quickly get very important information from the structures of the students´ concept maps. He can find out whether the students understand the logical side of the topic, whether they have a sufficient overview of the topic or are drowning in the information the meaning of which is unclear to them. A teacher can quickly find out if a student can really reflect on what he has learnt or whether he is just parroting some memorized phrases and the individual observations are mutually isolated. (Vasková, 2006)

The research group was created by the pupils from Year 4 at the Pavol Marcely primary school in Bratislava. 20 students took part in the research, 9 of them were boys and 11 were girls. When creating concept maps on the topics of a square, rectangle we chose unstructured concept mapping because we told the pupils only the key word (a triangle). The students´ task was to find other concepts which they could connect with the triangle (Janík, 2005).

THE PROCESS OF RESEARCH

The research took place in two consequent lessons. The procedure was following:

1. The first phase of the execution of the research was to prepare some exemplary concept maps.

2. Then there was finding out to what extent the pupils knew concept maps and what their experience with them was. We found out that the pupils were familiar with the concept maps as their teacher included them in the process of teaching not only

Math but other subjects as well. Therefore we concluded the concept maps would be on a very high level.

3. The preparation for working with the concept maps (the motivational part of the lesson) was based on sorting out similar shapes into groups according to their identical or alike characteristics. We prepared some cut out pictures of various objects which resembled specific geometric figures (e.g. road signs, a globe, a duvet, a tent...). Then the pupils one by one approached the objects and put a random picture into the relevant group. Thanks to this part of the lesson the pupils were appropriately motivated for the other part of the research which followed shortly.

4. Afterwards followed some shared work of the whole class. We used the exemplary concept map on the topic of *a circle.* We started with drawing a circle on the board together with the word „circle". Then together with the pupils we looked for objects that were of a similar shape. We wrote the correct objects on the board. Then we divided all the objects similar to a circle into various categories – a circle (a pizza, a cake, a placemat...), a ball (the Earth, an orange, a christmas ball...). Slowly we moved from the circle objects to the characteristics that all these objects had in common. The pupils needed a strong guidance from our side to correctly describe and define all the basic characteristics of a circle. The objects that were three-dimensional – a globe (the sun, planets, an onion, a ball ...) the students also included in this concept map of a circle. We presume that the students of this age do not quite differentiate between a drawn picture of a flat and a three-dimensional object and therefore they included the three-dimensional objects in the category of the circle objects because they viewed them as circles.

5. After working together as a class, we proceeded to working in groups of 4 pupils. We divided the class into 6 groups consisting of 2 boys and 2 girls. Each group got a task to create a concept map similar to the one we created earlier. 3 groups were to come up with a concept map for a rectangle and 3 groups for a square. During the group work we guided the pupils so that they understood and created the best possible maps. The group work did not cause any problems as this kind of activity was widely used with their teacher. After finishing the group stage we called out a representative from one of the groups with the *square* concept map to present the objects and characteristics included in their concept map. We wrote the correct ones on the board. Then the pupils from the other *square* concept map groups filled in their ideas. Even the pupils from the remaining groups participated in the discussion.

So we created one concept map on the concept *SQUARE* (fig.1) (for better visibility, we redraw it).

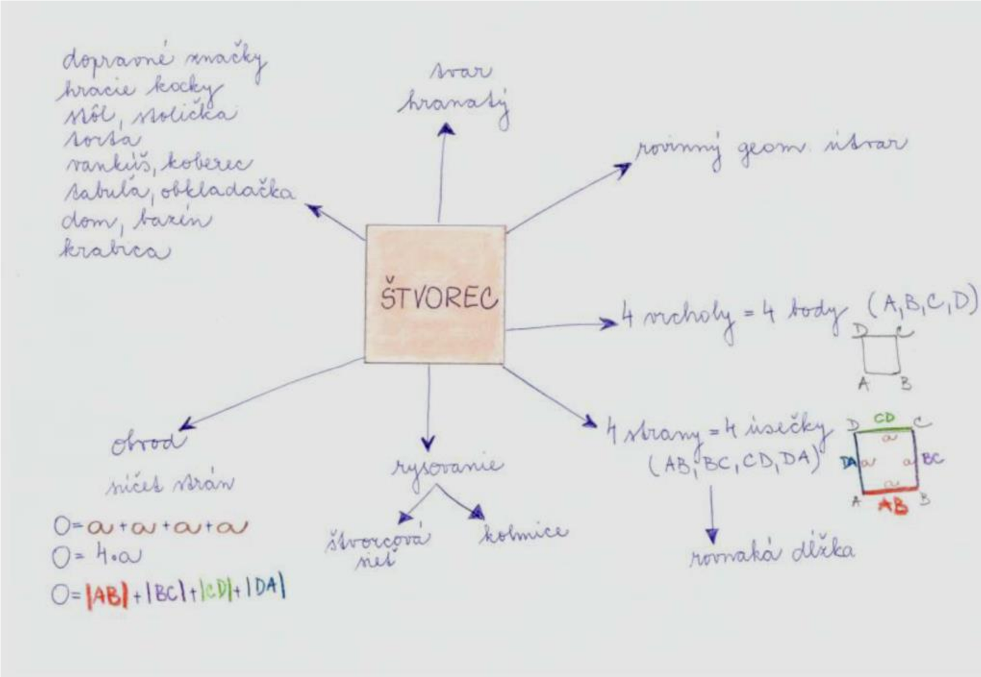

***Figure 1**: Concept map on the concept the SQUARE*

The concept map includes the notions:

- plane shape
- 4 points = 4 vertices (A,B,C,D)
- 4 sides = 4 segments (AB,BC,CD,DA) - are 4 sides are equal
- angled form
- drawing: square grid, perpendiculars
- perimeter/ the sum of the sides
- objects similar to a square (road signs, carpet, …)

We followed the same procedure with the *rectangle* concept map groups. Again the pupils from the other groups joined in with their ideas of objects and characteristics.

The common concept map of the class to concept *the RECTANGLE* is shown in figure 2.

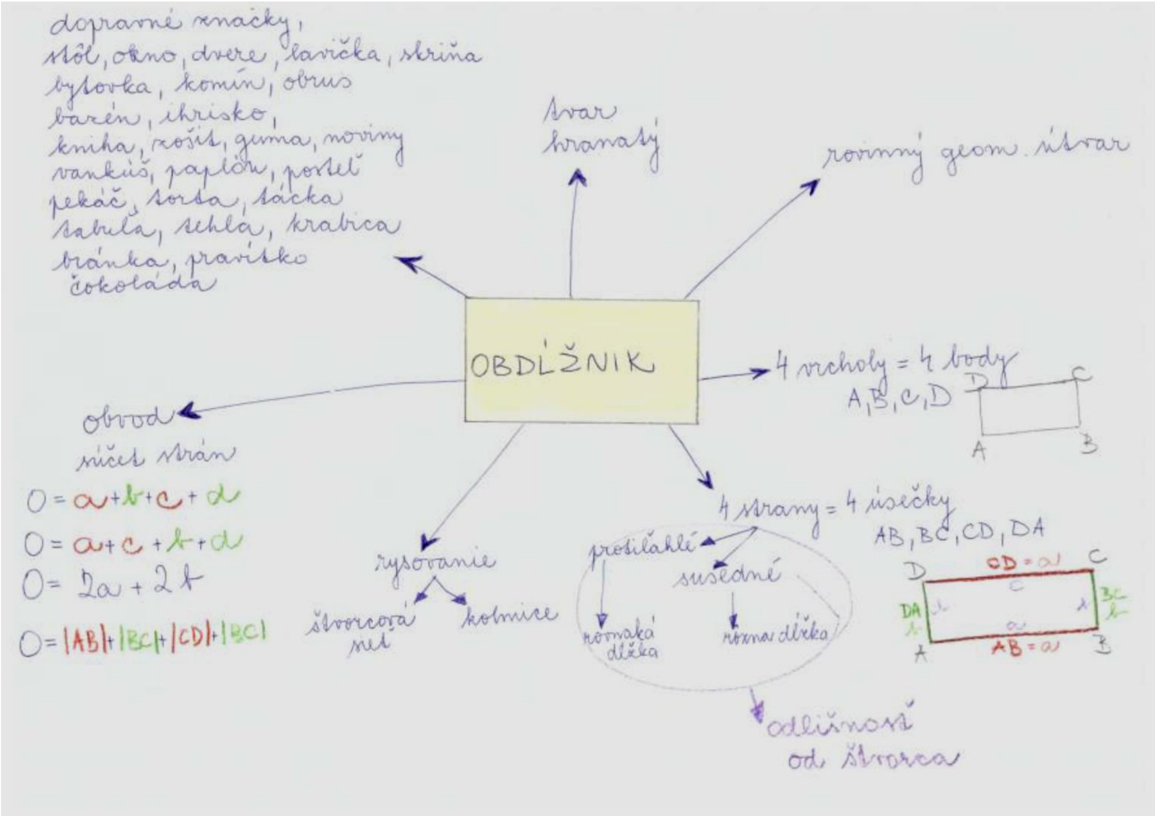

***Figure 2**: Concept map on tne concept the* RECTANGLE

The concept map includes the notions:

- plane shape;
- 4 points = 4 vertices (A,B,C,D);
- 4 sides = 4 segments (AB,BC,CD,DA) - Opposite sides are parallel and congruent – difference of square;
- angled form;
- drawing: square grid, perpendiculars;
- perimeter/ the sum of the sides;
- objects similar to a square (road signs, tablecloth, …)

At the end of this phase we showed the pupils the exemplary concept maps so that they themselves could see to what extent they managed to agree on the various objects and characteristics that we presumed they should know and include.

In this way, the pupils have created a tool to consolidate knowledge of the square and rectangle, and at the same time better understand the differences between the two geometric figures.

Conclusion

Concept maps organize knowledge very clearly into a comprehensive structure. Each of thematic curriculum is necessary summarize and the concept maps are suitable tool. They enable students to see all the newly acquired information. Concept maps could be prepared by the teacher himself, but much more effective they are when being prepared by pupils or students themselves only.

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Notes on Pythagorean Tetrahedron

Dušan Vallo[a*] – Viliam Ďuriš[a]

[a] *Department of Mathematics, Faculty of Natural Sciences, Constantine the Philosopher University in Nitra, Tr. A. Hlinku 1, SK - 949 74 Nitra*

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Abstract

Problem of existence of two classes of Pythagorean tetrahedrons is discussed in this paper. We derive characteristic equations of these tetrahedrons and we put the equations to the computer´s tests. Some models of the Pythagorean tetrahedrons are computing in this paper, too.

Keywords: Pythagorean Theorem, Pythagorean triple, tetrahedron, prism, characteristic tetrahedron.

Classification: G45

Introduction

Tetrahedron is a simplex in three dimensional space and such one is in some respects a more natural analogue for the triangle in two dimensional space. We consider a tetrahedron which edges form a Pythagorean triples representing three integer side lengths of right triangles as faces of the tetrahedron. [1] This point of view we develop in two ways.

At first, in according to [2] we re-investigate the existence of Pythagorean tetrahedrons which planar angles at one vertex are right. This type of the tetrahedron we name such Pythagorean tetrahedron of the 1st type.

One is called tetrahedron of the 2nd type if its planar angles at two different vertices are right. We also derive some characteristic equation with analogous parameters which determine an existence of tetrahedron of the 2nd type.

Note. The tetrahedrons of the 2nd type play a fundamental role in space filling. [3]

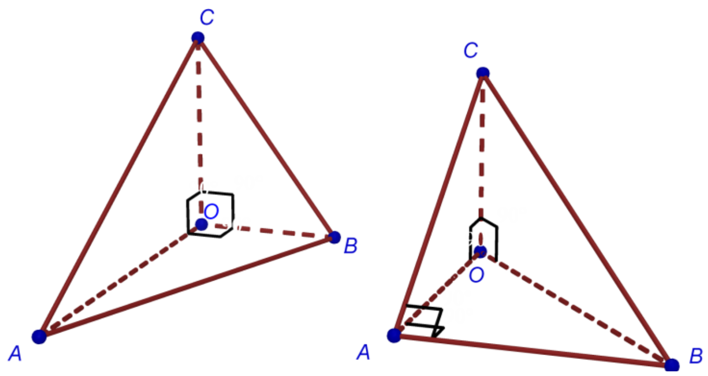

Figure 1*: Tetrahedrons with right planar angles*.

*Corresponding author: dvallo@ukf.sk
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Pythagorean tetrahedron – 1st type

We take up the terminology and derivation of the equation from [2]. We introduce it only on grounds of the understandable explanation for the reader.

We analogously consider the tetrahedron $ABCO$ which edges have labels $OA = a, OB = b, OC = c$, $AB = p$, $BC = q$, $AC = r$ and we also introduce parameters ε, η, ξ of the right-angled triangles ABO, BCO, ACO with planar right angles at vertex O.

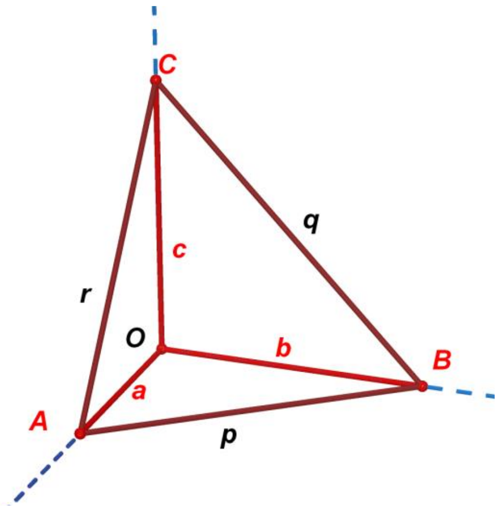

***Figure 2**: Tetrahedron of the first type. It has three right – angled triangles as the faces at vertex O.*

We define the parameters of the edges

$$\varepsilon = \frac{a}{b+p}, \ \mu = \frac{b}{c+q} \text{ and } \xi = \frac{c}{a+r}. \qquad (1)$$

It is evident that $0 < \varepsilon < 1$.

Using the Pythagorean Theorem it is also easy to prove that

a) $1 + \varepsilon^2 = 1 + \left(\frac{a}{b+p}\right)^2 = \ldots = \frac{2p}{b+p}$,

b) $1 - \varepsilon^2 = 1 - \left(\frac{a}{b+p}\right)^2 = \ldots = \frac{2b}{b+p}$,

c) $\frac{2\varepsilon}{1-\varepsilon^2} = \ldots = \frac{a}{b}$.

For the other parameters η, ξ hold analogous formulas. This implies that an equation for this type of Pythagorean tetrahedron is

$$\frac{2\varepsilon}{1-\varepsilon^2} \cdot \frac{2\eta}{1-\eta^2} \cdot \frac{2\xi}{1-\xi^2} = \frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{a} = 1. \qquad (2)$$

The equation (2) represents the test-criterion for computing of the Pythagorean tetrahedron.

If we put $\varepsilon = \dfrac{c_1}{m_1}$, $\mu = \dfrac{c_2}{m_2}$ and $\xi = \dfrac{c_3}{m_3}$,

where $c_i, m_i \in N, i = 1,2,3$, $1 \le c_i \le n$, $2 \le m_i \le n$ for arbitrary positive integer n, then holds true that for $OA = 1$ we can compute rational values of the edges $OB = \dfrac{1-\varepsilon^2}{2\varepsilon}$, $OC = OB \cdot \dfrac{1-\eta^2}{2\eta}$.

The reader can find the details and also results in article [2]. The author used the algorithm based on the idea (described above) to compile a computer program. The program ran for $n = 1,2,3,\dots,25$ and the hardware constructed on vacuum tubes generated 11 different Pythagorean tetrahedrons. From the reason that the calculation has been carried out 29 years ago, we have compiled a similar program and we have obtained the results in Table 1.

***Table 1**: Pythagorean tetrahedron – 1st type for n=1,2,3, …, 25.*

	a	b	c	p	q	r
1	1100	1155	1008	1595	1533	1492
2	252	240	275	348	365	373
3	720	132	85	732	157	725
4	1584	187	1020	1595	1037	1884
5	240	44	117	244	125	267
6	880	429	2340	979	2379	2500
7	231	792	160	825	808	281
8	480	140	693	500	707	843
9	3536	11220	2925	11764	11595	4589
10	5796	528	6325	5820	6347	8579
11	1008	1100	12075	1492	12125	12117
12	**780**	**2475**	**2992**	**2595**	**3883**	**3092**

If we compare the results with [2], the last result is a new Pythagorean tetrahedron in the list. This is a benefit of this section in our paper.

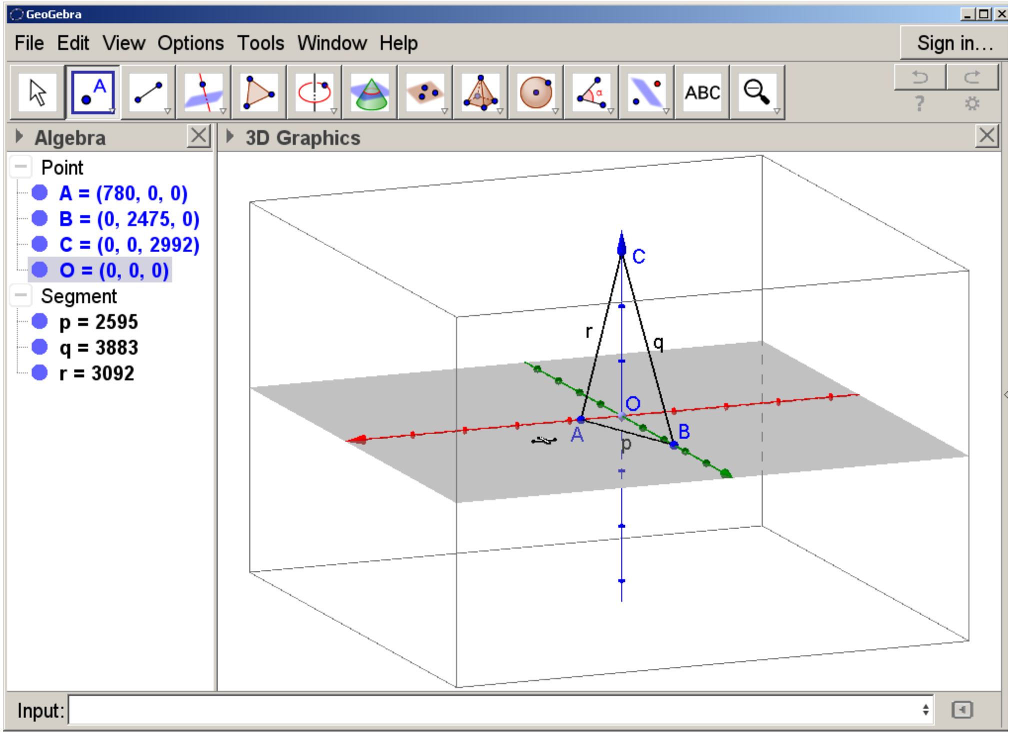

***Figure 3**: The Geogebra sketch of thelast Pythagorean tetrahedron in the list.*

Pythagorean tetrahedron – 2nd type

By analogy to previous case we consider the tetrahedron $ABCO$ which edges have labels $OA=a, OB=b, OC=c$, $AB=p$, $BC=q$, $AC=r$ (see Figure 2).

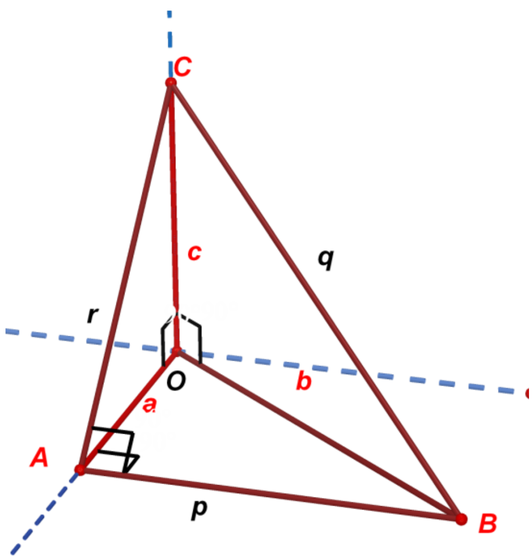

***Figure 4**: Tetrahedron of the second type. It has four right- angles triangles as the faces.*

In this position of the vertices we need not re-define the parameters of the edges and holds (1). Using the Pythagorean Theorem we derive

$$\frac{2\varepsilon}{1+\varepsilon^2}=\frac{a}{b},\ \frac{2\eta}{1-\eta^2}=\frac{b}{c}\ \text{ and }\ \frac{2\xi}{1-\xi^2}=\frac{c}{a}. \qquad (3)$$

This implies that an equation for the 2nd type of Pythagorean tetrahedron is

$$\frac{2\varepsilon}{1+\varepsilon^2}\cdot\frac{2\eta}{1-\eta^2}\cdot\frac{2\xi}{1-\xi^2}=1. \qquad (4)$$

The equation (4) represents the test-criterion for computing of tetrahedron in similar way such as Pythagorean tetrahedron of the 1st type. We have compiled a computer program. The program ran for $n=50$ and we have obtained the results in Table 2.

***Table 2**: Pythagorean tetrahedron – 2nd type for n=1,2,3, …, 50.*

	ε	**μ**	**ξ**	**a**	**b**	**c**	**p**	**q**	**r**
1	**1/13**	**4/5**	**9/17**	104	680	153	672	697	185
2	**1/9**	**6/7**	**13/41**	756	3444	533	3360	3485	925
3	**1/21**	**6/13**	**17/19**	252	2652	2261	2640	3485	2275
4	**1/13**	**9/17**	**4/5**	117	765	520	756	925	533
5	**1/18**	**10/11**	**13/35**	1584	14300	1365	14212	14365	2091
9	**1/21**	**17/19**	**6/13**	399	4199	468	4180	4225	615

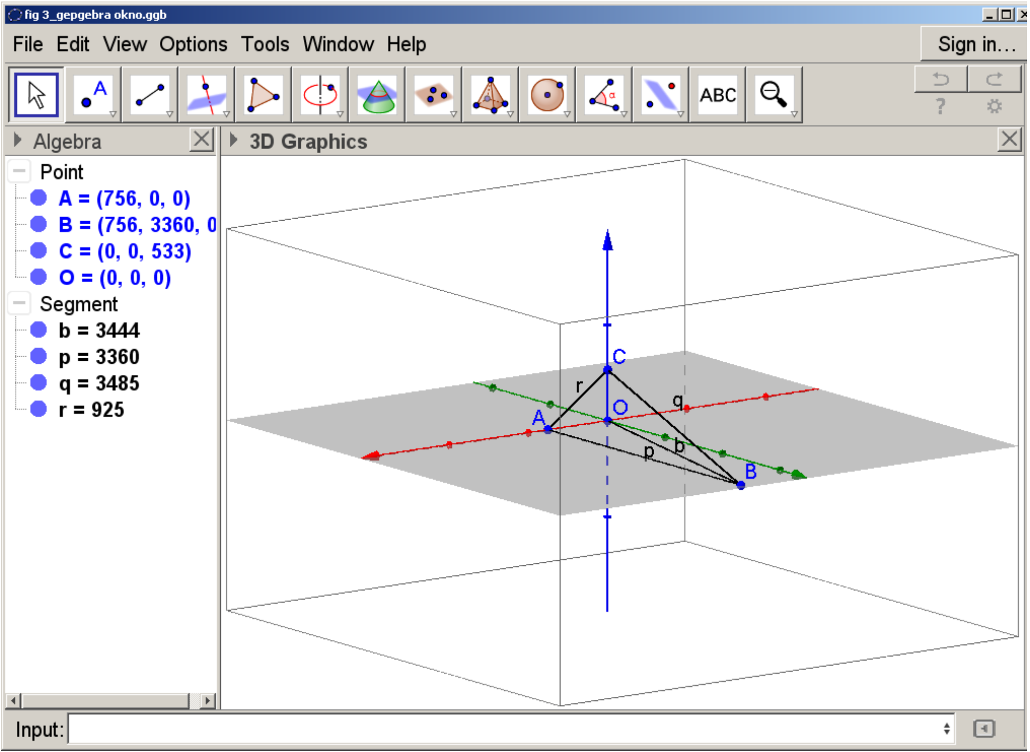

***Figure 5**: The Geogebra sketch of the Pythagorean tetrahedron of the 2nd type.*

Discussion

The idea of this paper comes out from the original paper [2]. Some inquisitiveness leads us to the verification of the results. Using a modern hardware, we have replenished one tetrahedron. The derivation and the computation of the Pythagorean tetrahedron of the 2nd type follows from the analogy.

The reader can find a very interesting problem in [2]. The Pythagorean tetrahedrons of the 1st type can be considered like the grounds of the rectangular parallelepiped in which every triple of edges or planar diagonals form Pythagorean triples, but the solid diagonals have irrational lengths in all cases! Does exist such parallelepiped? This problem is still open.

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Comparison of Two Research Tools Measuring Attitudes towards Mathematics

Peter Vankúš[a*] – Ivana Ochodničanová[a]

[a] *Department of Algebra, Geometry and Didactics of Mathematics, Faculty of Mathematics, Physics and Informatics, Comenius University in Bratislava, Mlynská dolina, SK - 842 48 Bratislava*

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Abstract

Our paper is comparing selected results from two research tools measuring attitudes towards mathematics. Both tools are questionnaires created to measure various areas of the complex system of pupils' attitudes. In our paper we compare the results on pupils' liking of mathematics found out by the first questionnaire with the results in the same area obtained by the second questionnaire. This comparison we realize on the sample of 154 pupils in the lower secondary education (10–15 years old). The statistical analysis of the data showed that both questionnaires give coherent outputs.

Keywords: attitudes, mathematics, research tools, questionnaire, attitudes towards mathematics.

Classification: C20

Introduction

The attitudes toward mathematics are important elements those are influencing the results of mathematics education. Therefore a lot of researchers are dealing with this important topic. Proper research tools for measuring various areas of the attitudes are needed to ensure the quality of the research in this topic. In our paper we study two research tools, the questionnaires. We compare the outputs those gave these questionnaires applied on the same research sample, to extract information on the correlations between results on the pupils' liking of mathematics found out by the first and by the second questionnaire.

Used questionnaires

The importance of pupils' attitudes towards mathematics is supported by the opinion, believed to be true in scientific and teacher communities which states that pupils learn more effectively and they are more interested in the mathematics lesson and are performing better if they have positive attitudes towards mathematics (Ma & Kishor, 1997). Therefore a lot of researches are done to study closely the system of pupils' attitudes towards mathematics. Very important tools in these researches are various questionnaires those are the tools to measure the areas of this system.

In our paper we deal with two such questionnaires. The first one was created based on the surveys on pupils attitudes towards mathematics performed in Slovakia. In this study that was part of the comparative research study led by Jose Diego-Mantecon and Paul Andrews

[*]Corresponding author: peter.vankus@gmail.com
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from the University of Cambridge was used the modified Mathematics Related Beliefs Questionnaire (De Corte & Op't Eynde, 2002). This questionnaire was designed to find out the compatibility of its use in various European states, specifically in England, Spain and Slovakia (Andrews, Diego-Mantecón, Op 't Eynde, & Sayers, 2007; Andrews, Diego-Mantecón, Vankúš, Op 't Eynde, & Conway, 2008; Andrews, Diego-Mantecón, Vankúš, Op 't Eynde, & Sayers, 2011).

Our first questionnaire that was used in the study described in this paper was created based on our analysis of the Slovak results of mentioned survey and also on our other studies that we performed in this field of the research (Vankúš & Kubicová, 2010; Vankúš &Kubicová, 2012). The questionnaire consists of 16 items and measures 4 areas: the liking of the mathematics, beliefs on the usefulness of mathematics, pupils' mathematics self-beliefs, and self-evaluation of pupils' effort in the mathematics. The complete questionnaire was published in our work (Vankúš, 2014, p. 134–135) that is also accessible on the webpage www.comae.sk/efektivnost.pdf. In this article we will work just with the items of this questionnaire dealing with the area the liking of the mathematics. These items, as well as the rest of the questionnaire, were 6 scale Likert type items. The items are included in the table 1.

Table 1: *Items from the first questionnaire*

I am fond of mathematics.
Mathematics makes me happy.
To learn mathematics is pleasure for me.
I like mathematics.

The second questionnaire that we study in this paper was used in our previous research (Vankúš, 2006; Vankúš, 2007). The questionnaire was created based on the sample questionnaire from the work by Mager (1984). The questionnaire has 4 items with the selection of answers and one item with possibility of the free answer. All the items are focused on the area of the liking of the mathematics. The questionnaire was published in our work (Vankúš, 2014, p. 128–129).

Research

The comparison of the questionnaires was done on the sample of 154 pupils in the lower secondary education (10–15 years old). The pupils were from two classes in the grades 5–8 and from one class in the grade 9. The research was done in the state school in Bratislava. The scoring of the used questionnaires was done based on the method described in already mentioned work (Vankúš, 2014, p. 130 and p. 138). The results are presented in the table 2. The results are done separately for each grade. The reason for this is the fact, found out by the previous researches that the attitudes towards mathematics are changing with the age of pupils (Ma & Kishor, 1997; Vankúš &Kubicová, 2012).

***Table 2**: Research results*

5th grade (10–11 years old) ***n = 43***	**First questionnaire**	**Second questionnaire**
Mean	17,21	15,77
Standard deviation	4,72	4,30
Normality test (Shapiro-Wilk, *W*)	0,94 (not normal)	0,97 (normal)
6th grade (11–12 years old) ***n = 25***	**First questionnaire**	**Second questionnaire**
Mean	15,44	14,76
Standard deviation	3,37	3,98
Normality test (Shapiro-Wilk, *W*)	0,94 (normal)	0,95 (normal)
7th grade (12–13 years old) ***n = 26***	**First questionnaire**	**Second questionnaire**
Mean	12,08	10,85
Standard deviation	5,87	5,79
Normality test (Shapiro-Wilk, *W*)	0,91 (not normal)	0,92 (not normal)
8th grade (13–14 years old) ***n = 37***	**First questionnaire**	**Second questionnaire**
Mean	13,00	13,03
Standard deviation	5,02	4,35
Normality test (Shapiro-Wilk, *W*)	0,97 (normal)	0,96 (normal)
9th grade (14–15 years old) ***n = 23***	**First questionnaire**	**Second questionnaire**
Mean	11,87	10,91
Standard deviation	4,18	4,81
Normality test (Shapiro-Wilk, *W*)	0,97 (normal)	0,93 (not normal)

Based on the normality Shapiro-Wilk test stated in the table 2 the results from the questionnaires are not all normally distributed ($p < 0{,}10$), so we will use both parametric and nonparametric tests. We will now compare the results of the questionnaires by the parametric Student's t-test (for normally distributed data) and nonparametric Mann-Whitney U-Test (for all data). The results from these tests are in the table 3.

***Table 3**: Statistical comparison*

5th grade		
Mann-Whitney U-Test	$U = 748{,}0$	$p = 0{,}13$
6th grade		
Student's t-test	$t = 0{,}65$	$p = 0{,}52$
Mann-Whitney U-Test	$U = 267{,}5$	$p = 0{,}39$
7th grade		
Mann-Whitney U-Test	$U = 293{,}0$	$p = 0{,}42$
8th grade		
Student's t-test	$t = 0{,}02$	$p = 0{,}98$
Mann-Whitney U-Test	$U = 682{,}0$	$p = 0{,}98$
9th grade		
Mann-Whitney U-Test	$U = 221{,}5$	$p = 0{,}35$

From the data in the table 3 we can see that results from both questionnaires are not statistically significantly different ($p < 0{,}10$). So we can say that our questionnaires give statistically coherent outputs. We can also compute the Pearson coefficients of the correlation, which are in table 4. They show moderate positive correlations, which means there is a tendency for high scores in the first questionnaire go with high scores in the second one (the same for the low ones).

***Table 4**: Pearson coefficients*

Grade	5th	6th	7th	8th	9th
Pearson coefficient	$R = 0{,}74$	$R = 0{,}80$	$R = 0{,}91$	$R = 0{,}84$	$R = 0{,}75$

Conclusion

In our paper we compared the results from our two questionnaires used to study pupils' attitudes towards mathematics. In both questionnaires we focused on the area of the liking of mathematics. We have done statistical comparison of the results obtained from 154 pupils. The statistical data show that our questionnaires give coherent outputs those are in correlation. This implies that the questionnaires are properly built. Further studies on the bigger samples are still needed to verify the validity and reliability of these research tools. That will be beneficial for the next studies in the field of attitudes towards mathematics.

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Infinite Series in Physics

Marek Varga[a*] – Zuzana Naštická[a]

[a] *Department of Mathematics, Faculty of Natural Sciences, Constantine the Philosopher University in Nitra, Tr. A. Hlinku 1, SK - 949 74 Nitra*

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Abstract

In calculus lectures and seminars students frequently call for demonstrations of usefulness of a new term or proposition. Tasks about tangent or instantaneous velocity are quite legendary, showing the need for derivatives. Similarly, pictures of inscribed or circumscribed rectangles, leading to the definition of Riemann definite integral, are quite popular. In the submitted contribution we aim to demonstrate the usefulness of infinite functional series (power, Taylor), which come in useful for common calculations in physics, and also for formulation of new discoveries and theories in physics.

Keywords: Infinite series, motion of body, magnetic force, Einstein equation.

Classification: M15

Introduction

In mathematics classes with cast iron regularity teachers encounter the most favourite student question – "what is it good for?" And, this does not depend on the level of education, whether at elementary or secondary schools, or at universities. A possible response (well, difficult to say if satisfactory for the questioner) is introduction of selected application tasks. In the following lines we introduce several applications of infinite (functional) series in physics.

Position of a body

Let us assume that the position of a mass point in time t_0 is expressed by coordinate $x(t_0)$. If this point is moving, we are interested in its position after certain period of time Δt passes. If it is a „sufficiently short" time interval, we can consider it to be a case of uniform motion – i. e. the velocity $v = \frac{dx}{dt}$ was constant. Then, we get

$$x(t_0 + \Delta t) = x(t_0) + v \cdot \Delta t \, .$$

In real conditions, however, uniform motion of a body is quite rare. A more accurate estimate of the new position can be, thus, obtained if we assume that it was a case of uniformly accelerated motion with constant acceleration $a = \frac{d^2x}{dt^2}$.

*Corresponding author: mvarga@ukf.sk
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For the new position of the mass point we have

$$x(t_0+\Delta t)=x(t_0)+v\cdot\Delta t+\frac{1}{2}a(\Delta t)^2 .$$

Next, we can say that also the uniformly accelerated motion is quite a specific case, and the acceleration is not necessarily constant; there can be a kind of „acceleration of acceleration". Let us label this quantity with symbol α; then $\alpha=\frac{d^3x}{dt^3}$. For the new position of the mass point we would get even more accurate value

$$x(t_0+\Delta t)=x(t_0)+v\cdot\Delta t+\frac{1}{2}a(\Delta t)^2+\frac{1}{6}\alpha(\Delta t)^3.$$

The preceding assumptions imply that after infinitely many "improvements" the real position of the mass point would be

$$x(t_0+\Delta t)=x(t_0)+k_1\cdot\Delta t+k_2(\Delta t)^2+k_3(\Delta t)^3+\dots;$$

where for each coefficient k_n applies

$$k_n=\frac{d^nx}{dt^n};\ \text{where}\ \ n=1,2,3\dots$$

Finally, we can write

$$x(t_0+\Delta t)=x(t_0)+\left.\frac{dx}{dt}\right|_{t=t_0}\cdot\Delta t+\left.\frac{d^2x}{dt^2}\right|_{t=t_0}\cdot(\Delta t)^2+\left.\frac{d^3x}{dt^3}\right|_{t=t_0}\cdot(\Delta t)^3+\dots .$$

From the mathematical point of view, we can say that function $x(t)$ is written in form of a power series, or in form of a Taylor series with midpoint t_0. The natural part of the mathematical solution would be determining the domain of convergence of the series, i. e. the set of all values t for which the sum of the series is a real number.

Magnetic force

Let there be an electric conductor, with current density τ. Recall that the charge of the conductor is determined by positive (protons) and negative particles (electrons), therefore $\tau=\tau^++\tau^-$. If no electric current flows in the conductor, its total charge is zero, i.e. the body is electrically neutral. If current starts flowing in the conductor, its positive charge does not move, i. e. positive current density preserves the initial value, so $\tau^+=\tau_0$. However, electrons start moving within the conductor, and in terms of the theory of special relativity for negative current density we get $\tau^-=-\tau_0\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}$, where v is the velocity of electrons and c is the speed of light.

Here, mathematics enters the game, in form of the power (Taylor) series expansion for a function, as it holds good that

$$(1+x)^m = 1 + \binom{m}{1} x + \binom{m}{2} x^2 + \dots,$$

where $|x| < 1$. In our case this requirement is met, as $\frac{v}{c} < 1$, since nothing moves faster than light in vacuum. Moreover, already the third and all the following terms of the series are insignificantly small in comparison with the first and the second terms, therefore we can effectively work with the approximation

$$(1-x)^{-\frac{1}{2}} \approx 1 + \frac{1}{2} x .$$

The total current density in the conductor is then

$$\tau = \tau^+ + \tau^- = \tau_0 - \tau_0 \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} = \tau_0 \left[1 - \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}\right] \approx -\frac{\tau_0}{2} \cdot \frac{v^2}{c^2} .$$

Thus, the physical root of the situation can be handled in an "accelerated manner" – the force exerted by the field, which is formed by the moving electrons, on the charge with magnitude Q positioned in distance r, is

$$F_m = -QE = -Q \frac{\tau}{2\pi\varepsilon r} \approx Q \frac{\tau_0 v}{4\pi\varepsilon r c^2} v = QvB ;$$

where B is the magnetic induction. Force F_m, which is actually the relativistic consequence of the existence of the electric force, is referred to as the magnetic force.

Einstein's equation

According to the theory of special relativity, the mass m of a body is not a constant quantity, but its value depends on the velocity v of the body. Naturally, this is only recognizable if the velocity is comparable with speed of light in vacuum c. If the rest mass of the body ($v = 0\, m/s$) is m_0, then the mass of the body moving at speed v is

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} .$$

Here, we encounter the series from the above lines again, namely

$$(1+x)^m = \sum_{n=0}^{\infty} \binom{m}{n} x^n = 1 + \binom{m}{1} x + \binom{m}{2} x^2 + \dots = 1 + mx + \frac{m(m-1)}{2} x^2 + \dots .$$

Then, we get

$$(1-x)^{-\frac{1}{2}} = 1+\frac{1}{2}x+\frac{3}{8}x^2+\dots, \text{ resp. } \left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} = 1+\frac{1}{2}\frac{v^2}{c^2}+\frac{3}{8}\frac{v^4}{c^4}+\dots.$$

As $v < c$, suffice it to use approximation $\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} \approx 1+\frac{1}{2}\frac{v^2}{c^2}$, leading to

$$m = m_0\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} \approx m_0\left(1+\frac{1}{2}\frac{v^2}{c^2}\right).$$

Having multiplied by c^2 we get

$$mc^2 = m_0c^2+\frac{1}{2}m_0v^2 = E_0+E_k.$$

On the right side there is the sum of rest and kinetic energy of the body, i. e. its total energy E, and thus we get the famous Einstein's equation

$$E = mc^2.$$

Conclusion

Mathematics is the queen of all sciences. Leonardo da Vinci even dared to say that if an exploration does not use mathematical methods, it cannot be considered a science. Applications of mathematics are surely present in many branches, perhaps the most appreciably in physics. Issues of functional series, power series, and Taylor series do not belong to introductory topics within university calculus courses, they are quite demanding. Similarly, their uses in physics (although we tried to minimize the physical stuff in this contribution) do not belong to the easiest examples of applications. If nothing more, we at least managed to emphasize that the birth of several spectacular physical theories, such as description of magnetism, or theory of relativity, was accompanied by these mathematical theories.

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Mathematics Knowledge of Pre-service Teachers for Primary Education and Their Readiness for Practice

Kitti Vidermanová[a*] – Lucia Rumanová[a] – Júlia Záhorská[a]

[a] *Department of Mathematics, Faculty of Natural Sciences, Constantine the Philosopher University in Nitra, Tr. A. Hlinku 1, SK-949 74 Nitra*

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Abstract

This study is focused on knowledge of pre-service teachers for pre-school and primary education. Selected requirements for pupils' knowledge in the fifth year of their schooling are compared with pre-service teachers' knowledge in primary mathematics. As a research tool we applied the mathematical part of an admission test to eight-year grammar school which covers the curriculum of primary mathematics.

Keywords: Knowledge, pre-service teachers, primary education, mathematics.

Classification: D39

Introduction

At the first level of primary school teachers strongly influence how pupils succeed throughout their schooling. They need to teach pupils use mathematical thinking for solving practical problems in everyday situations; appropriate use of ICT for learning, identification of risks associated with the use of the Internet and other media; how to develop critical thinking when working with information; how to apply acquired knowledge of natural sciences and social sciences; to recognize problems at school and in their nearest environment, and finally, how to propose a solution according to their knowledge and experience. (Innovated *State Education Programme for Primary Education*, 2015, page 5) In Slovakia all teachers have to be university graduates and must have appropriate higher education, and pre-school teachers are not an exception. The study programme for pre-school and primary teachers is common in a three-year bachelor degree (called *Pre-school and Elementary Education*). The bachelor diploma authorizes the graduates work as pre-school teachers or tutors for school clubs. University graduates can continue in two-year master degree programme (*Teacher Training for Primary Education*). This programme has to be completed by receiving the teaching qualification (teacher training for primary school).

In primary educational level pupils should achieve the following objectives, focused on mathematics (Innovated State Education Programme for Primary Education – Appendix Mathematics, 2015):

- acquire basic mathematical concepts, knowledge, skills and practices stated in educational standards;

*Corresponding author: kvidermanova@ukf.sk
DOI: 10.17846/AMN.2015.1.1.160-167

- perform computations with integer numbers (up to 10,000);
- use fractions in the propaedeutic, preparatory level;
- identify and correctly name of the functional relationships between numbers,
- explore rules within given sequences and continue the sequences;
- orient in tables and charts,
- identify and construct geometric figures specified in educational standards,
- estimate and accurately measure lengths and also convert length units (mm, cm, dm, m, km);
- use mathematics as a tool for finding solutions to real life problems (including development of financial literacy);
- develop skills related to the process of learning; improve cognitive processes and mental operations;
- strengthen positive moral and volition-related characteristics (independence, decisiveness, endurance, tenacity, criticism, self-criticism, confidence in their own abilities and possibilities, systematic problem solving in personal and public contexts);
- develop key competences in social and communication spheres.

At primary school, pupils have *Mathematics* as a school subject one lesson almost every day. The primary mathematical curriculum is divided into five main parts: *Numbers, variable and number operations*; *Geometry and measurement*; *Relations, functions, tables, diagrams*; *Logic, reasoning, proofs; Combinatorics, probability, statistics*.

Pre-service teacher training programmes for pre-school and primary education at universities in Slovakia

One of the nowadays trends is the decreasing number of contact lessons at Slovak universities. This problem negatively impacts each study programme, e. g. decreasing of mathematical content or number of subjects in teacher training programmes for pre-school and primary education.

While future teachers for primary education study at university, they attend various subjects and courses with mathematical content.

We compared the bachelor and master study programmes from five universities in Slovakia. Below, we give an overview of the number of contact hours (Table 1) which cover various branches of mathematics and didactics of mathematics within these programmes.

The subject Didactics of mathematics is included in the master study programme, not earlier. Only one of these universities (J. Selye University) includes the subject Mathematics and Didactics of mathematics in bachelor study programme. The list of subjects for the study of different branches of mathematics contains subjects, such as Introduction to geometry; Mathematical imagination (methodology of its developing); Formation of geometrical imagination; Logic and Set theory; Arithmetic; Didactical games in Mathematics; Development of basic mathematical imagination; Manipulative geometry; Geometry; Primary mathematical education; Elementary Graph theory; Geometric modelling and spatial imagination; Integers, rational and real numbers; Functions and functional thinking; Phylogeny and ontogeny of

Integers; Mathematical literacy; Methodology of Problem solving; Mathematics and work with information; Workshops in mathematics; Mathematical seminars and competitions; Formation of mathematical concept of number, and Basics of mathematics; Amusing tasks.

Table 1: Overview of the number of hours in bachelor and master degree programme of five universities in Slovakia

University	Bachelor degree programme Pre-school and Elementary Education			Master degree programme Teacher Training for Primary Education		
	subjects			subjects		
	compulsory	compulsory-optional*	optional	compulsory	compulsory-optional	optional
Constantine the Philosopher University in Nitra	8	4	0	7	6	0
Catholic University in Ruzomberok	5	1	0	5	2	1
Trnava University in Trnava	3	4	0	12	10	0
Matej Bel University Banská Bystrica	8	4	4	4	2	4
J. Selye University	12	2	0	This university does not provide the master study programme.		

According to the data in the overview we can state that different universities have assigned different number of mathematical preparation lessons and courses for future primary school teachers.

In our opinion higher quality of teacher training is the necessary precondition for quality pupils´ education at various levels of education. Latest national and international measurements of pupils' mathematical knowledge indicate a declining trend. Reducing the number of contact lessons at university would cause further decrease of students' mathematical knowledge as well as their readiness for practice. Therefore, it is important to consider adjustments of the teacher training curriculum in university courses oriented to the primary education.

*Not directly compulsory subject, but students have to select several subjects from a group of compulsory-optional subjects.

The study

For our study we decided to use the mathematical part of admission test to eight-year grammar school in Nitra, Golianova Street 68. Eight-year grammar schools in Slovakia provide lower and upper secondary education. Pupils in the fifth year of their schooling can apply for admission to this study. The main reason for the choice of such test is that the entire primary mathematical curriculum is covered in the test tasks.

We divided the participants into three groups. The first group consisted of 77 pupils who wrote the admission test. The average age of these pupils was 11 years. These pupils were in the end of the fifth year of their schooling. The second group consisted of 88 pre-service teachers, university students of Pre-school and Elementary Education study programme, who were in the end of the second year of a three-year bachelor study programme. These students had completed several mathematical subjects oriented to basics of Arithmetic, Algebra and Geometry which were focused only on mathematical apparatus, without any Didactics of mathematics. The third group consisted of 62 students of Teacher Training for Primary Education study programme who were in the end of the second year of a two-year master study programme. These students had completed all mathematical subjects, including problem solving strategies and didactics of mathematics. It is important to remark that these students were just preparing for state exams and they should start their professional career three months later. We expected them to be familiar with mathematical cognizance and also know how to teach primary mathematics by using appropriate teaching methods and strategies.

The test consisted of two parts. The first part included four tasks: three numeric calculation tasks (two focused on curriculum topic *Numbers, variable and number operations,* one on *Geometry and measurement*) and one construction task to draw a line, circle, square or rectangle, perpendicular line and parallel line. The second part of the test included seven multiple choice tasks: three tasks covering curriculum topic *Numbers, variable and number operations,* one covering *Geometry and measurement* with focus on spatial geometry, one focused on *Relations, functions, tables, diagrams*, one on *Logic, reasoning, proofs,* and one covering *Combinatorics, probability, statistics*.

We have set the following research questions:

Is the average score of the test for university students higher than the average score of pupils in the end of the fifth year of their schooling?

Have the students at the end of their master study better results from the test than students in bachelor study?

Results and discussion

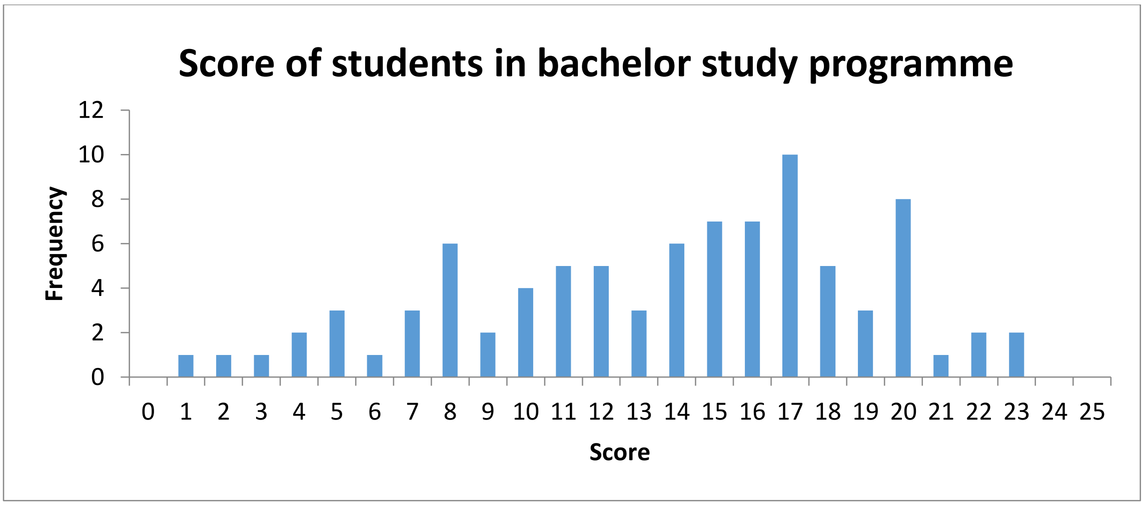
The maximum score of the test is 25 points. The average score of each group of participants is presented in Table 2. The supplemented statistical characteristics are included, too.

The average score of university students is not higher than the average score of pupils in the fifth year of their schooling. Students in master study programme have higher average score than pupils, but the difference in not as significant as we expected.

Table 2

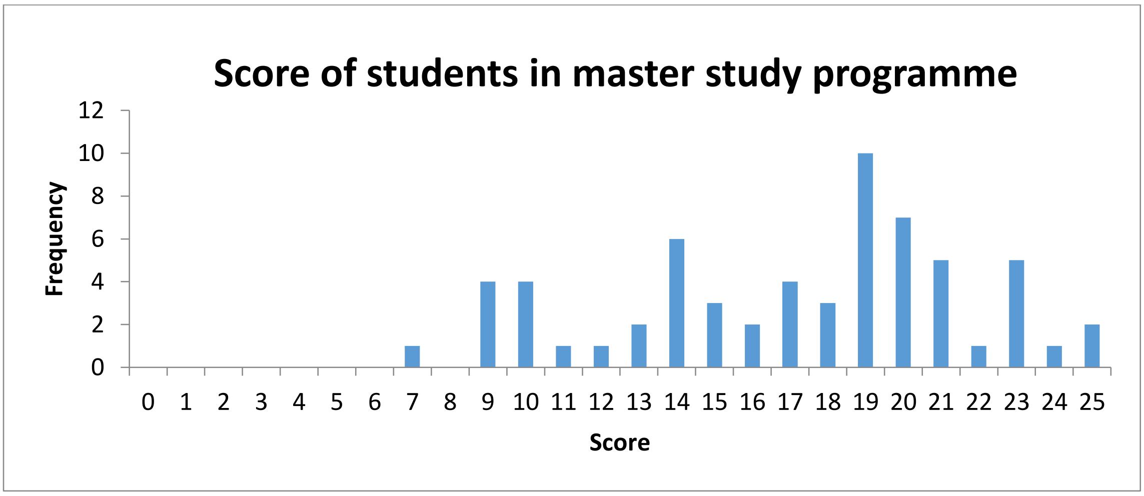
	Pupils who wrote the admission test Pupils in the fifth year of their schooling	Students of Pre-school and Elementary Education Bachelor study programme	Students of Teacher Training of Primary Education Master study programme
Number of participants	77	88	62
Average score	15,17	13,69	17,11
Percentage	60,68 %	54,77%	68,45%
Median	15	14	19
Mode	14	17	19

Graph 1: Histogram of score of students in Pre-school and Elementary Education programme



Students in bachelor study programme have smaller average score than the pupils, but their mode is higher than the pupils' mode. The average score of students in bachelor degree is reduced due to students with extremely low count of points (there were 18 students with 1-8 points, see Graph 1).

Graph 2: Histogram of score of students in Teacher Training for Primary Education programme



Students at the end of their master study have better results from the test than students in bachelor degree, which is an expected result, although we expected much better results (about 90 %) of the master degree students.

Furthermore, the students used solving strategies which are not adequate to the level of primary mathematics. They used formulas and equations with unknown, rule of three and decimal numbers which are not a part of the primary maths curriculum.

Despite their weak results students of the bachelor study programme were grateful for "adjusting a mirror" of their knowledge.

We introduce also the transcript of a dialogue among the students of Pre-school and Elementary Education study programme and the teachers. The dialogue occurred after evaluation of the test:

S_1: *How can I teach pupils these tasks if I don´t know them myself?*

T: Don´t worry about that. If you don´t know it now, you have another three years to the end of Teacher training of primary education programme. Use this time effectively.

Other student:

S_2: At lower secondary schools we were taught to use equations with unknown, rule of three, decimal numbers, fractions and percentages. Now, couldn't we use these strategies in solving task or problems?

T: You can use the whole mathematical apparatus which you have in solutions of tasks or problems during the preparation to education. However, with children you can use only strategies adequate to the curriculum of primary mathematics.

The approach of another student was surprising:

S_3: I will use methods and strategies described in the methodological guide to textbooks. There will be everything what I will need.

T: Not all textbook authors write a methodological guide to textbooks. What will you do if there does not exist any guide to textbooks used in the class?

S3: (Silent)

Comparison of the scores - topics of the primary mathematics

Furthermore, we were interested in which curriculum topic the students had the weakest results. Table 3 shows the percentage of students in each curriculum topic.

After testing the students said that the easiest task was the combinatorial problem. They had difficulties with finding regularity; how to calculate the perimeter of rectangle, and with the task related to spatial geometry.

Table 3

Curriculum topic	Bachelor study programme Pre-school and Elementary Education	Master study programme Teacher Training of Primary Education
Numbers, variable and number operations	57,44 %	72,43 %
Geometry and Measurement	47,28 %	58,06 %
Relations, functions, tables, diagrams	60,23 %	84,68 %
Logic, reasoning, proofs	67,05 %	79,03 %
Combinatorics, probability, statistics	81,82 %	85,48 %

The following examples are the tasks with the lowest success rate in each curriculum topic.

Numbers, variable and number operations

Hedgehog Pichliač and his wife hedgehog Pichličica had a wedding day and they invited 35 hedgehog guests. Among the guests there were adult hedgehogs and young hedgehogs, too. There were 15 less young hedgehogs than adult hedgehogs. How many young hedgehogs were invited to the wedding party?

Geometry and Measurement

Peter glued together 10 cubes in one solid, as shown in Figure 1. Then he coloured the solid with blue colour. After dismantling the solid, how many cubes have three blue faces?

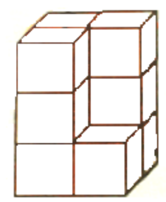


Figure 1

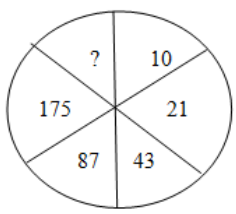
Relations, functions, tables, diagrams

A mole sews trousers for itself and friends. The mole needs 28 dm of cloth for eight trousers. How many trousers can the mole sew from 42 dm of cloth?

A) 6 B) 10 C) 12 D) 15

Logic, reasoning, proofs

Fill in the circle with numbers following the regularity:



A) 351 B) 346 C) 236 D) 216

Combinatorics, probability, statistics

Mark can get to school through three different streets. He always uses a different street on his way from school than he uses on his way to school. How many days can he walk to and from school without repeating his way?

Conclusion

A lot of studies suggest that prospective elementary teachers need to study more mathematics (in Mewborn, 2001). Ball & Wilson (1990) presented the same students' questions as we introduced: *How can I teach if I don't know why myself? We haven't seen these things before and I don't know where I was supposed to learn them – in high school? Or middle school?*

In our opinion, mathematics learning needs to be accompanied by learning adequate problem solving strategies in teacher training study programmes for pre-school and primary education. As we mention in the introduction of this paper, teachers in the first level of primary school often affect how pupils succeed throughout their schooling and within that also he/she affects their attitudes to mathematics. Teachers prepare pupils for various mathematical competitions. Last but not least, teachers at primary school should be able to prepare pupils for admission test similar to the one we used the research tool.

The difference between score of students in bachelor and master degree shows that inclusion of the mathematical subject (including didactical subjects oriented on problem solving strategies) into teacher training programme has its justification. Nevertheless, the score of master students is not satisfactory. Therefore, it is important to consider curriculum adjustment of the teacher training study programme for primary education.

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Deficiencies in Geometric Language Register and Knowledge of Pre-service Teachers for Primary Education

Zuzana Vitézová[a] – Zuzana Naštická[a*]

[a] *Department of Mathematics, Faculty of Natural Sciences, Constantine the Philosopher University in Nitra, Tr. A. Hlinku 1, SK - 949 74 Nitra*

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Abstract

Children adopt the language register of their close relatives and of their teachers. In terms of mathematics education, the most important language and information input is exposed to children by their teachers at primary educational level. The submitted contribution aims to investigate the deficiencies in fundamental geometric knowledge of university students within Teacher Training Master Programme for Primary Education at Constantine the Philosopher University in Nitra, Slovakia. The research was conducted in April, 2015, within a 90-minute long university seminar at the Department of Mathematics. Altogether 50 students of the master programme filled in a worksheet focused on selected fundamental geometric terms. The worksheet served as a warm-up and a follow-up activity for the main activity of the seminar when students constructed 'skeleton' models of selected solid figures. The presented results are based on observation and content analysis of the worksheets.

Keywords: prism, pyramid, skeleton of a solid figure, errors

Classification: D60, D70

Introduction

During their lifetime people learn mainly through experience. Thus, it is natural that children need to try and touch everything. After children start attending school, they have to learn more and more through reading and writing. For some children this change can be difficult. Teachers can make this transition easier for pupils, e. g. by using appropriate educational aids (Gabajová & Vankúš, 2011). In addition, it is very important that children learn to express themselves properly, already within the primary educational level. This is also related to using accurate register and terms of the specific school subjects. Teachers play crucial role in achieving this objective, since pupils adopt their language devices and register. That is why we selected the below described activities, i. e. making skeletons of solid figures with the use of drinking straws and string, for university students within Teacher Training Master Programme for Primary Education (further only 'primary teacher trainees').

Objectives and research tools

We can distinguish two types of objectives that we set to achieve by the activities. The first group consists of the educational goals. We aimed to help the teacher trainees revise their knowledge about right-regular prisms and pyramids. We focused on planar figures which

*Corresponding author: zuzana.nasticka@ukf.sk
DOI: 10.17846/AMN.2015.1.1.168-175

form faces of the solid figures, and on number of edges, faces and vertices, and on names of the solids and planar shapes. Another educational goal was to introduce a simple way of creating an educational aid, which could be useful in their future teaching practice.

The research objective was to identify the errors that students made when filling in the table (see Fig. 2) and when defining several mathematical terms. The terms we selected for the activity are fundamental in geometry, and we believe primary teacher trainees should be able to explain their meanings accurately.

The tool we used for data collection is a two-page worksheet, with selected geometric terms which the teacher trainees were asked to define on one page, and a table about selected solids they were asked to fill in on the other page (Fig. 2). After the teacher trainees filled in and handed in their worksheets, the content analysis of the worksheets was carried out.

Prisms and pyramids in mathematics education

According to the National Educational Programme ISCED 1, solids are firstly introduced to pupils in the fourth year of elementary school, when the focus is primarily on cubes, building cube constructions and drawing their plans (ISCED 1). In the fifth year also cuboid is introduced, though only in a propaedeutic form; prisms are not introduced before the eighth year, and pyramids in the ninth year of elementary school (ISCED 2).

Nevertheless, it is inevitable that teachers in primary educational level use proper mathematical terms when referring to the fundamental objects, solids and their constituents, so that pupils are exposed to correct information from the very beginning. What pupils learn at the primary level is what they come with to further classes.

Material aids

One of the fundamental requirements on mathematics education is the formation of thought images and notions through visual perception and objective knowledge. In other words, it is the necessity to uphold the principle of clearness and illustration which represents the pupils’ need to acquire new knowledge through concrete sensory perception of phenomena and objects. Observation of real objects, drawings and photographs, thought images elicited by speaking and listening can serve this purpose (Šedivý, 2006), as well as material educational aids for specific curriculum. Driensky and Hrmo (2004) characterize educational aids as material tools which are direct vehicles of information that pupils are supposed to acquire. An educational aid can provide educational content through a technical device or directly (Driensky & Hrmo, 2004).

Gábor et al. (1989) distinguish three main groups of educational aids: a) demonstrative and frontal, b) audio-visual, c) material. From the perspective of didactics of mathematics educational aids can be divided into two groups, according to the target group of people they are meant for, either for teachers, or for pupils.

In the activity we designed for the teacher trainees haptic (tactile) aids were used. Despite material aids are often technically inaccurate, they are very influential in the initial abstraction stage of mental ontogenesis, leading to formation of thought images representing fundamental geometric notions (Vallo, Rumanová, Vidermanová, & Barcíková, 2013). Students – teacher trainees were provided with material for production of ‘skeleton’ models of solid figures. After they managed to make the models, they could touch them and observe them in order to be able to fill in the table in the worksheet (Fig. 2).

Activities with students

Within the university seminar, which lasted for 90 minutes, students did two main activities. Altogether 50 primary teacher trainees took part in two such seminars. Below the activities and the results are described in more detail.

The first activity

In the initial part of the seminar students were asked to think of and write down in their own words definition of several geometric terms, namely planar figure, solid figure, vertex, side, edge, face and base of a solid figure. Students wrote their suggestions in the worksheets. Before analysing the student definitions, let us introduce definitions proposed by mathematicians.

A **planar figure** can be defined as a subset of the two-dimensional Euclidean plane which can be described with the use of points, lines and circles, sets, logic and arithmetic symbols, and the notion of distance (Hejný et al., 1989). We expected more simple formulation by students, e. g. that a planar figure is a geometric shape in a two-dimensional plane.

A **solid figure** is a part of space bounded by a surface which does not intersect itself (Pavlič, 2001). A solid is a closed three-dimensional geometric figure (Clapham & Nicholson, 2009).

A **vertex** of a polygon is any point at which two sides of the polygon meet; a vertex of a polyhedron is any point at which (at least) three edges of the polyhedron meet (Alexander & Koeberlein, 2011).

A **side** is one of the lines joining two adjacent vertices in a polygon (Clapham & Nicholson, 2009). It is a line segment which forms a part of the boundary of a planar figure.

An **edge** of a polyhedron is any line segment that joins two consecutive vertices of the polyhedron (Alexander & Koeberlein, 2011). It is the common side of two boundary polygons of a polyhedron.

A **face** of a polyhedron is any one of the polygons that lies in a plane determined by the vertices of the polyhedron; it includes base(s) and lateral faces of prisms and pyramids (Alexander & Koeberlein, 2011). It is a boundary polygon of the polyhedron.

The **base** of a solid figure is the face of a solid figure to which an altitude is drawn (Alexander & Koeberlein, 2011).

As we have already mentioned, we did not expect such accurate definitions of the notions. Students were asked to explain the notions in their own words. We were aware of the fact that it is very difficult to be accurate in definitions. Also experts in didactics of mathematics warn that it is rare to find such a fundamental mathematical term which could be given an absolutely exact definition (Fischer, Malle, & Bürger, 1992). It is said that accuracy of mathematical language is relative (Hejný, et al., 1988).

Expected errors

According to Hejný et al. (1990) there are three types of bond disruption within the process of notion formation, namely: (i) wrong thought images are assigned to words and symbols, (ii) no thought images are assigned to words and symbols, (iii) no language expression has developed (within the individual brain) for thoughts and thought images.

We expected that in the solutions of our students all the three types would occur, and that the most frequent errors would be (i) and (iii). Our expectations proved to be correct.

Results

As we could foresee, all involved students described the base of a solid as “the part of the solid which it stands on”. They did not care much about why and when (for what purposes) the notion of base is necessary in mathematics.

We further expected that students would define a planar figure as “a shape lying in a plane, or drawn on the paper”. However, some of the student definitions were positively surprising, such as “a planar figure is a set of points in a plane, for example a triangle” or “it is a shape in a plane, determined by at least three line segments which meet at three points”.

Most of the students described a solid figure as a “three-dimensional shape with certain volume and surface area”. Some students also stated that “it is a closed region of the space”, and that “unlike a planar figure, a solid figure is in the space and is observed in 3D version”.

Students described a vertex as “the point at which two or more lines meet”, or “the place where two or more line segments intersect each other”. These and other similar student definitions of vertex met our expectations.

When asking students for their definitions of terms *side*, *edge* and *face,* our main focus was on determining if they knew when it is appropriate to talk about sides and when about edges, and what is the main difference between them. Several students stated that a side “is a line segment joining vertices of a shape”, yet, many students did not state if the shape is two- or three-dimensional. Some students referred to a side as a line, or they related it to both planar figures and solids (in the Slovak language, *side* is used only when talking about planar figures, *edge* and *face* only in relation to solids). For instance, “a side is the line connecting two boundary points of a solid or a planar shape”, or “it is a line which forms the shape, for example in a square all four sides are perpendicular to each other; it is related to planar figures”.

As for the term *edge,* several students stated quite accurate definitions, such as “an edge is a line segment, bounded by two vertices, connecting two adjacent faces; it is related to solid figures”. A less accurate, but revealing much valuable information about the learner perception, is the following definition of an edge, stating that “it is a line at which as if two faces were broken”.

Attempting to define the notion of *face,* students stated that “it is a planar shape which bounds the perimeter of the solid”, or that “it is a set of points in a plane, which is bounded by the edges of the solid; faces are planar figures which the solid consists of”.

The educational goal of this activity was to make students aware of the main differences and relations between sides, edges and faces. As evidenced above, several students understand the notions and managed to explain them using their language register quite accurately. Yet, many of the students do not really understand the notions, and thus cannot use them properly.

The second activity

In the second part of the seminar students worked in twos or threes. Their task was to construct ‘skeleton models’ of eleven solids with the use of drinking straws and string. The selected solids were regular tetrahedron, octahedron, dodecahedron, cube, right-regular hexagonal prism and pyramid, right-regular pentagonal prism and pyramid, right-regular four-sided pyramid, right-regular triangular prism, and regular icosahedron. Students were not asked to construct the model of icosahedron, since it would take them too much time. Except for icosahedrons, students managed to construct models of all ten solids (Fig. 1).

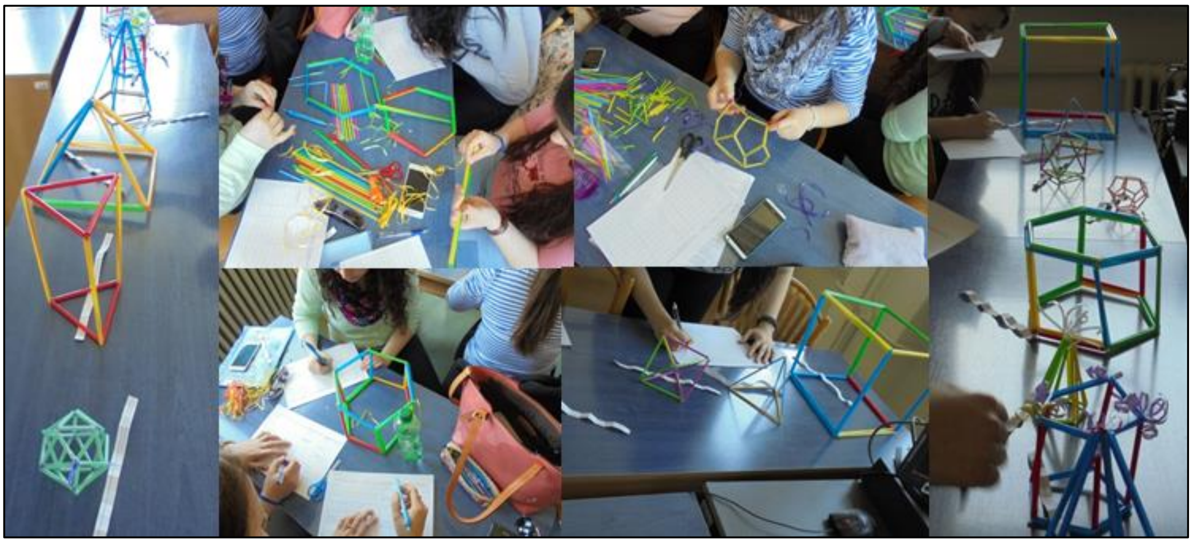

Figure 1

A follow-up task was to fill in a simple table related to the constructed solids. Students were asked to state the names of the solids, the number of their faces, edges and vertices, the shapes of the faces, and the number of the bases. The correct solution is shown in Fig. 2, the light orange shading is applied on those solids which we focus on in this contribution.

	Name of solid figure	Number of			What planar shapes are the faces of this solid?	Does this solid have any bases? If yes, how many?
		faces	edges	vertices		
1.	Cube	6	12	8	Square	No
2.	Regular tetrahedron	4	6	4	Equilateral triangle	No
3.	Regular octahedron	8	12	6	Equilateral triangle	No
4.	Regular dodecahedron	12	30	20	Regular pentagon	No
5.	Right-regular hexagonal prism	8	18	12	Regular hexagon, rectangle/square	Yes, 2
6.	Right-regular hexagonal pyramid	7	12	7	Regular hexagon, isosceles triangle	Yes, 1
7.	Right-regular pentagonal prism	7	15	10	Regular pentagon, rectangle/square	Yes, 2
8.	Right-regular pentagonal pyramid	6	10	6	Regular pentagon, isosceles triangle	Yes, 1
9.	Right square pyramid (right-regular four-sided pyramid)	5	8	5	Square, isosceles triangle	Yes, 1
10.	Right-regular triangular prism	5	9	6	Equilateral triangle, rectangle/square	Yes, 2
11.	Regular icosahedron	20	30	12	Equilateral triangle	No

Figure 2

Results obtained from the tables filled-in by students

When analysing the student tables, we focused on right-regular prisms and pyramids. We aimed to explore the level and quality of knowledge that the involved primary teacher trainees possess about planar and solid figures and their constituents. We also wanted to find out what problems may emerge within this topic.

We considered the student solutions correct also in cases when they did not specify the prisms and pyramids as right-regular or the polygons as regular, or when the shapes of bases were stated in an inappropriate column of the table. The least problematic solid turned out to be cube. Out of 50 teacher trainees 49 managed to fill in all the cube-related cells correctly. The remaining solution was incorrect, for the student stated the name of the solid to be tetrahedron instead of cube.

Many problems occurred with respect to right-regular four-sided pyramid. In total seven students incorrectly stated its name to be right-regular *three*-sided pyramid. Perhaps, when filling in the table, students expected the solids in line 9 and 10 to be of the same type, or they could get puzzled by the model of the solid itself. We had not expected such errors, since prisms and pyramids were explicitly named in the instruction for the task (see Fig. 3, blue shading).

1.	With the use of drinking straws and string, make a skeleton of a solid figure the surface area of which consists of six congruent square faces.
2.	With the use of drinking straws and string, make a skeleton of a solid figure the surface area of which consists of four congruent equilateral triangles.
3.	With the use of drinking straws and string, make a skeleton of a solid figure the surface area of which consists of eight congruent equilateral triangles. Four edges meet at each vertex of this solid.
4.	With the use of drinking straws and string, make a skeleton of a solid figure the surface area of which consists of twelve congruent regular pentagons. Three edges meet at each vertex.
5.	With the use of drinking straws and string, make a skeleton of a right-regular hexagonal prism.
6.	With the use of drinking straws and string, make a skeleton of a right-regular hexagonal pyramid.
7.	With the use of drinking straws and string, make a skeleton of a right-regular pentagonal prism.
8.	With the use of drinking straws and string, make a skeleton of a right-regular pentagonal pyramid.
9.	With the use of drinking straws and string, make a skeleton of a right-regular four-sided pyramid.
10.	With the use of drinking straws and string, make a skeleton of a right-regular triangular prism.
11.	With the use of drinking straws and string, make a skeleton of a solid figure the surface area of which consists of twenty congruent equilateral triangles. Five edges meet at each vertex.

Figure 3

For each of the solids, except for the cube, there were students who failed to state the correct number of faces; in some cases it was evident that they were not sure if also bases are faces, although they managed to identify the shapes of faces, including the bases.

Altogether 34 teacher trainees failed to identify the faces of the right-regular four-sided pyramid, i. e. isosceles triangles. They incorrectly specified it as equilateral. The remaining students just did not specify the type of triangle.

Problems during the activities

With respect to the way the students approached the task to define the selected geometric terms in their own words, we believed they can easily distinguish between planar and solid figures. In most cases it is obvious they have certain comprehension about the meaning of the terms, yet they cannot express it due to lack of comprehension of other important related notions, such as line, segment, altitude, incidence, intersection point. Having analysed the filled-in tables, we conclude that not all the involved teacher trainees can distinguish between planar and solid figures and use mathematical terms correctly, as evidenced in the following statements of the students: "A face is a segment connecting two points; the segments form a three-dimensional solid and are labelled with lower-case letters." "An edge is a point where two segments meet; the points form a solid figure and are labelled with capital letters." "A side is a segment which bounds a drawn solid; it is labelled with lower-case letters with respect to the opposite vertex. A solid is two-dimensional." "A vertex is the point where two lines/half-lines, or two arcs etc. intersect." "A side is a line determined by two points in a shape." Also, one student asked during the seminar if "a face is not the same as a side".

The most problematic seemed to be the issue of bases. Some students did not count bases among faces. Also, they did not realize the point of talking about bases in a solid, e. g. they did not realize there is no point in talking about bases in a cube. Some of the students stated that a cube has six bases, just because all the six faces are congruent.

Working in groups also brought some complications. In one group students almost did not manage to construct the model of a right-regular five-sided pyramid. One member of the group thought that the base face should be a pentagon; yet, the two remaining students persuaded him that it should be a square. The group realized their mistake later in the seminar and re-made the model.

Although the following issue is not related to prisms and pyramids, it has much to do with notions and thought images in mathematics. Having analysed the filled-in tables, we found out that many of the students cannot distinguish between some planar and solid figures. Namely, some students stated pentahedrons to be faces of dodecahedron, instead of pentagons; similarly, some students referred to regular tetrahedron as equilateral triangle.

Conclusion

We conclude that the involved primary teacher trainees have not fully acquired fundamental geometric terms. They do not distinguish between planar and solid figures, or their constituents. This can cause severe problems in mathematics education of pupils at primary level, who might adopt incorrect register from their teachers, which will require re-education later on.

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Verification of Quality of Some Tasks in Tests of KEGA Grant Projects

Peter Vrábel*

Department of Mathematics, Faculty of Natural Sciences, Constantine the Philosopher University in Nitra, Tr. A. Hlinku 1, SK - 949 74 Nitra

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Abstract

The four problems with eight tasks of KEGA grant project (Increasing the key mathematical competencies I, II) tests are specified by Bloom´s taxonomy and their quality is verified in this paper.

Keywords: Test task, Bloom´s taxonomy, reliability.

Classification: 97*C40, 97D10, 97U20, 97U30*

Introduction

Not very impressing results of Slovak pupils´ assessment in international PISA measurements and new state educational program encourage mathematical education from elementary school to be more oriented on development of knowledge and skills in solving of real-life situation problems, which need to be imposed and formulated. Therefore it is necessary to create collections of good and interesting practical problems, proven and taught in elementary school conditions. The Slovak Ministry of Education grant projects No. 3/7001/09 and 015 UKF-4/2012 (*Increasing the key mathematical competencies-alternative teaching programs in mathematics for elementary schools in terms of objectives of new state educational program and in terms of elevating of mathematical competences accordance impact PISA I, II*) was oriented on working out such methodical materials which will advance elevating of mathematical competencies of pupils of elementary schools. The methodical materials consisted of tasks with real-life context and methodical instructions for teachers (Fulier [4], 2014). The effectivity of the didactic materials was consequently verified in the mathematical education process. The level of mathematical competencies of pupils was investigated by posttests at the conclusion of the pedagogical experiment. We verify the quality of eight tasks from the posttests and we specify them by Bloom´s taxonomy (Bloom [2], 1956; Anderson [1], 2001). We observe reliability mainly. These tasks belong to topical sphere *Relations*, *functions, tables and diagrams.* This topical sphere complements and naturally intersects with the topical sphere *Numbers, variable and number operations* which merges into solving equations and inequations by means of propaedeutics of variable quantity.

Eight tasks from posttests

*Corresponding author: pvrabel@ukf.sk
DOI: 10.17846/AMN.2015.1.1.176-182

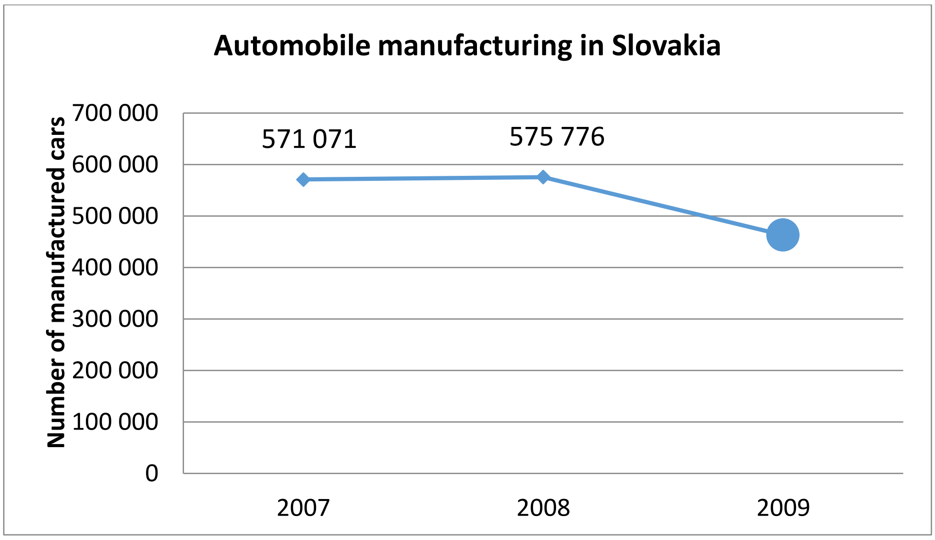
The level of mathematical competencies of pupils was investigated by a posttest in every grade 5 -8. The posttest for every grade contained 6 problems. One problem from them was from topical sphere Relations, functions, tables and diagrams. Each problem consisted of two tasks. The tasks of this sphere are more or less concentrated on ability understanding reading in the work with various tables and diagrams and also on formation of skills to use natural numbers and decimal numbers in the description of real situations in grades 5 and 6. The introduction of fractions in grade 7 ends in applicative branch: percent, calculus of interest, representation of a component element and a percent count by appropriate diagrams. Finally a pupil shall acquire abilities and knowledge to create mathematical models of simple real situations by utilization of literae as variables. These mathematical models are simplest linear equations or linear functions.
We used the following tasks from topical sphere Relations … :

Grade 5

How many cars did manufacture automobile factories?

Three automobile factories act in Slovakia. Information about number of manufactured cars is in the following table and graph:

Automobile factory	Year		
	2007	**2008**	**2009**
KIA	160 301	201 507	151 032
PEUGEOT CITROËN	204 060	186 397	205 030
VOLKSWAGEN	206 710	187 872	107 078



Task 1. Find out which automobile factory manufactured the most cars in year 2008. Round their number to thousands.

Task 2. Fill the missing number for the year 2009 into the graph.

Grade 6

Telephone lines

The price list of telephone lines T-com for the programmes **Indoors Comfort** and **Indoors Mini** is described in the following table. Free minutes are used minutes for which a customer does not pay.

Programmes of telephone lines		**Indoors Comfort**	**Indoors Mini**
Classification fee		1 €	1 €
Monthly fee		11.90 €	6.73 €
Free minutes		0	30
Prices of domestic calls for one minute		**Indoors Comfort**	**Indoors Mini**
Residential calls	Peak period	0.0757 €	0.1513 €
	Off-peak period	0	0.0956 €
	Weekend	0	0.0797 €
Intercity calls	Peak period	0.1554 €	0.3266 €
	Off-peak period	0	0.1513 €
	Weekend	0	0.1195 €

Task 1. Fill up the following table for the programme **Indoors Mini.**

INDOORS MINI	RESIDENTIAL CALLS Off-peak period: 30 min. Used free min..: 20	INTERCITY CALLS Peak period: 30 min. Used free min.: 10
Charged sum	$10 \cdot 0.0956 = 0.956$ (€)	

Task 2. The monthly payment for a telephone line is calculated for each programme by applying the following formula: *monthly fee + fee for used minutes.*
Fill in the following table and calculate the monthly payment of programme **Indoors Comfort.**

INDOORS COMFORT	RESIDENTIAL CALLS Peak period:40 min.	INTERCITY CALLS Off-peak period: 30 min.	Monthly fee	Total
Price charged				

Grade 7
Term deposit

The family of Thrifty´s open 1 year term deposit in Slovak Savings Bank at 1st July 2011 to the amount of 26 000 € with interest rate 4%. They planned to use the saved money to buy a new car KIA. Automobile store provided discount of 800 € from the beginning of year 2012 until May 2012.

Task 1. Of how many percent decreased the car price after the discount if it originally cost 16 000 €?

Task 2. Interest is taxed by 20 %. Interest is not paid in case of premature withdrawal of the money from the term deposit. Would it be for family Thrifty better to buy the car in May or in July?

Grade 8
Consumption of car

Cars have different average fuel consumption in city and outside city. For example Skoda Fabia has declared average fuel consumption 5.6 litres on 100 km outside city and 9.6 litres on 100 km in city.

Task 1. Fill in the following table with approximated fuel consumption in litres if we drive with Fabia only in city.

km	50	100	150	215	x
consumption		9.6			

Task 2. Mr. Rosina rides 1000 km a month with Fabia where from the 1000 km he rides outside the city x km. Denote by *S* whole fuel consumption in litres for a month. How much litres fuel *S* is drained depending on x? Write *S* as an expression with variable x.

Pupil results of the investigated tasks

The descriptive statistics of scores of the introduced tasks are in the Table 1 and the percent distributions of pupil fruitfulness of the investigated tasks are in Figure 1.

Table 1: *Descriptive statistics*

	Number of pupils	Mean	Standard Deviation	Minimum	Maximum	Mode	Median
5 T1	887	1.37	0.778	0	2	2	2
5 T2	887	1.63	1.425	0	3	3	2
6 T1	737	0.746	0.891	0	2	0	0
6 T2	737	0.614	0.822	0	2	0	0
7 T1	593	0.58	0.495	0	1	1	1
7 T2	593	1.12	1.516	0	4	0	0
8 T1	544	1.158	0.706	0	2	1	1
8 T2	544	0.478	0.977	0	3	0	0

It is evident that the tasks 6 T1 and 6 T2 were too difficult for the pupils of grade 6 (mode and median equal 0).

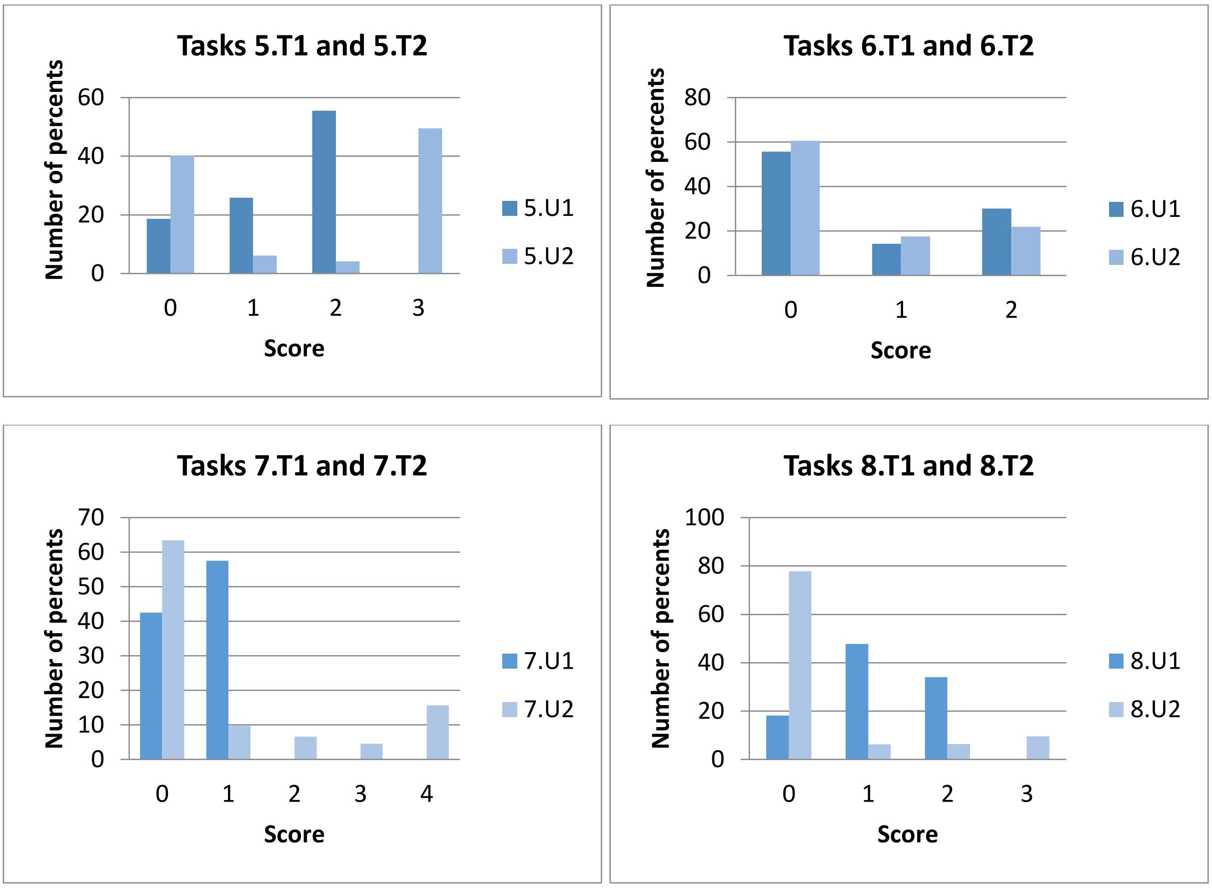


***Figure 1:** Number   percent of pupils which achieved score 0 – 4 for tasks 5 T1 – 8 T2*

Verification of quality of eight tasks

The posttest for every grade aimed at advanced level abilities how´s the memory acquisition of curriculum. Pupil abilities to solve tasks with real-life context were tested.

All tasks investigated in this paper are open-ended. We classify the tasks with regard to two dimensional Bloom´s revised taxonomy (Table 2).

***Table 2:** Bloom´s revised taxonomy of tasks*

	The knowledge dimension	The cognitive dimension
5 T1	conceptual knowledge	understand
5 T2	procedural knowledge	analyse
6 T1	conceptual knowledge	apply
6 T2	procedural knowledge	analyse
7 T1	conceptual knowledge	apply
7 T2	procedural knowledge	evaluate
8 T1	conceptual knowledge	analyse
8 T2	procedural knowledge	evaluate

The tasks 5 T1, 6 T1, 7 T1, 8 T1 are classified to the dimension of conceptual knowledge; the task T1 is classified to the cognitive dimension understand; the tasks 6 T1 and 7 T1 are classified to the dimension apply; the task 8 T1 is classified to the dimension analyse. The tasks 5 T2, 6 T2, 7 T2, 8 T2 are classified to the dimension of procedural knowledge; the tasks 5 T2 and 6 T2 are classified to the cognitive dimension analyse; the tasks 7 T2 and 8 T2 are classified to the dimension evaluate.

The reliabilities of posttests measured by Cronbach`s alpha and test reliabilities if one of investigated tasks is deleted are in Table 3.

Table 3: *Cronbach´s alpha and Cronbach´s alpha if item deleted*

Year	Scale Mean	Cronbach's Alpha	Scale Mean if Item Deleted		Cronbach's Alpha if Item Deleted	
			T1	T2	T1	T2
5.	17.49	0.823	16.12	15.86	0.807	0.807
6.	12.120	0.851	11.372	11.504	0.839	0.840
7.	10.39	0.862	9.82	9.40	0.858	0.843
8.	13.230	0.873	12.069	12.748	0.863	0.870

All posttests are reliable (all Cronbach`s alpha are greater than 0.82) and no one from investigated tasks does not decrease the test reliability.

Finally we introduce the pupil results of the investigated tasks in the experimental group and the control group (Fulier [3], 2014).

Table 4: *Number of students*

Grade	Group E	Group C	Total
5.	429	458	887
6.	385	352	737
7.	294	299	593
8.	221	323	544

Table 5: *Means and Chi-square Test of independence of score variable and group variable results*

	5.T1	5.T2	6.T1	6.T2	7.T1	7.T2	8.T1	8.T2
Mean E	1.417	1.837	0.938	0.813	0.687	1.415	1.452	0.905
Mean C	1.325	1.435	0.534	0.395	0.465	0.579	0.957	0.186
Mean total	1.370	1.837	0.745	0.613	0.575	0.993	1.158	0.478
Chi-square value	3.601	18.480	38.808	47.971	29.948	49.425	73.868	105.383
Degrees of freedom	2	3	2	2	1	4	2	3
p-value	0.165	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Rejection of independence	Not	Yes	Yes	Yes	Yes	Yes	Yes	Yes

The score means are greater in the experimental group of students for all tasks and p-values in the Table 5 confirm that the pupil's scores are group dependent for all tasks excepting task 5 T1. That means the task 5 T1 was not significantly difficult for the control group of students than for the experimental group of pupils. All others tasks significantly discriminated in between groups.

Conclusion

The investigated tasks confirmed the didactic efficiency of new learning materials which were created within the framework of the KEGA 3/7001/09 project and the KEGA 015 UKF-4/2012 project.

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The Principles of the New Mathematics Textbooks in Hungary

Gergely Wintsche*

Mathematics Teaching and Education Centre, Faculty of Natural Sciences, Eötvös Loránd University, Pázmány Péter sétány 1/c, 1117 Budapest, Hungary

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Abstract

In the following I give a short summary about the planning process and development of the new mathematics textbooks in Hungary. I am going to show that the basic theoretical principles are applied in the new textbooks. I bring examples of good practices for theoretical problems such as for problem solving tests. The first feedbacks from teachers confirm that we are on a good path however we have to meet their need for a lot of ready to use easy tasks for the everyday teaching practice. They were really enthusiastic about the games and put them well into practice but we need to work on the literacy competency skill of the students and teachers.

Keywords: Textbooks, mathematics, teaching.

Classification: U23

Introduction

The development of new curriculum and textbooks meeting the 21st century requirements are high priority in Hungary. We started to develop new textbooks and other learning tools for the age group 6-18 with the support of Social Renewal Operational Programme in 2014. In the first year the mathematics, literature and grammar, history and common natural science pilot books were developed for the grade 1, 2, 5, 6, and 9, 10 students. This offer will expand with the number of books available for grade 3, 7 and 11. The physics, chemistry, geography and biology textbooks are currently in the publishing process in 2015. The series will be completed with the set for grade 4, 8 and 12 in the autumn of 2016.

In the following I describe the curriculum development process which takes three years. In the first year the authors and the creative editor write the pilot version of the textbooks. In the second year the books are tested in practice by thousands of students and at least fifty teachers. We collect systematically the teacher feedbacks week by week. These feedbacks are then handled by independent educational experts who compile an analysis and an assessment. In the third year we utilize their results to reconstruct and rewrite the respective parts of the pilot textbooks. The volumes will get their final form only in the third year.

In spite of the former published textbooks every pilot textbooks are available and downloadable from the internet without any restriction.
(see http://etananyag.ofi.hu/tantargyak/matematika)

*Corresponding author: wintsche@cs.elte.hu
DOI: 10.17846/AMN.2015.1.1.183-187

It was an essential aspect to rethink the possibility of the didactic rules of the digital media and build connections to the printed books as well. An unimpeachable and inevitable part of the project is the development of the national education portal, where in the future everybody can access the connected digital teaching tools.

In this article I focus on how we achieve our main goals in the 5-8 grade mathematics textbooks. I demonstrate several good examples from the textbooks and in the end I will cite some teacher opinions.

Theoretical background

The textbook is a specialized book which supports the learning and teaching of mathematics. The regular place of usage is in the schools and homes, thats is why the most important users are the teachers and students. This is the origin of the didactical triangle (Rezat 2008a, p. 177, Schoenfeld, 2012).

Valverde (Valverde et al. 2012) completed it with the mathematical knowledge to the textbook−student−teacher didactical triangle as the fourth vertex and they got a three dimensional triangle, a tetrahedron. See figure 1.

We do believe that this three dimensional tetrahedron model describes the connections in the system more precisely.

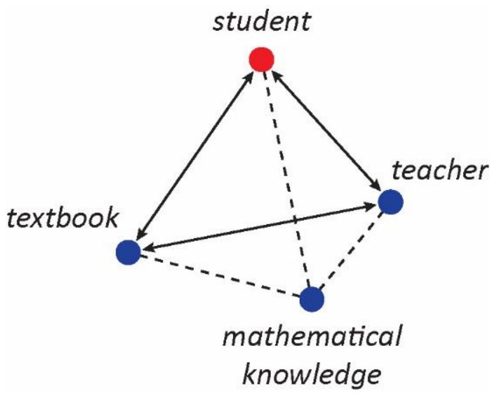

***Figure 1**: The didactical triangle or tetrahedron*

Naturally there are several players besides teachers and students who would like to influence the textbook development according to their preferences. Let us take a look at them and their wishes.

mathematician:	Be precise and mathematically correct.
teacher:	Provide help for the everyday challenges.
didactic expert:	Let it be easy to understand for students and educate them to think.
student:	It shall be interesting and enjoyable.
government	Suit the aim of the state.
parents:	It should be financially available and easy to handle.

Main principles

In the Hungarian and international practice the interest of the mathematicians and teachers are first priority in the textbook development process. Moreover we have to respect additional viewpoints of other stakeholders as well. Unfortunately in Hungary the student needs were at the end of the line. However when the Varga Tamás teaching method was launched in the 1970's we were in the lead of the student focused mathematics teaching. Due to several reasons it was lately neglected.

In the beginning of the new textbook process one of our most important decision was to respect the Hungarian traditions and put the students into the focus of development. Parallely we directed our attention to meet the requirements of the teachers as well. This is the main reason why there are several games, research tasks, and problems that intend to improve literacy competency. We created several methodological recommendations for the teachers. For example if the students shall work on the problems in pairs or groups. These suggestions are of course not obligatory to follow but give a good direction how the tasks can be used in the most efficient way.

There is an international expectation that the students shall acquire an applicable set of knowledge by finishing the school years. To reflect this in the textbooks is only possible if the teachers can recognize and utilize these goals, consequently our main task is to provide inservice teacher training.

The socialization of children to student life starts approximately at the age of 6. If they are in the primary school confronted with mathematics only as summation, subtraction, multiplication and division we shall not expect that in the upper grades they will evolve and start to think beyond. We need to start to teach them thinking from the first class on!

Based on this the authors and the creative editor (me) are strongly committed that the mathematics textbooks need to serve problem solving, practice oriented and realistic mathematics teaching at the first sight. We need to build it in a frame which is liked by the students and acceptable by the teachers. At the same time we emphasize that based on the personality of the teacher and their class, it is the teacher's responsibility to find the most efficient way that the students like, understand and learn the curriculum.

The pilot textbooks

Each chapter starts with a whole page graphic as a storyboard connected to the main topic of the chapter to grab the attention of the students and build an emotional connection. These short stories build on each other and form a story of a fantastic class excursion. See figure 2.

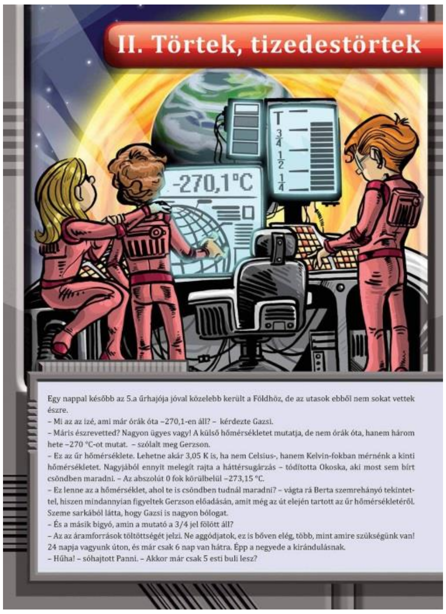

Figure 2*: The open page of the chapter of fractions from the book grade 5 (page 43)*

According to the first feedbacks this idea was liked by the students although some of them found them a little too long and caused confusion in the teachers in some cases.

Games and manipulating works

In each chapter we aimed to place games connected to the topic. Important to note that these are real games and not only tasks that are usually considered as games only by the mathematics teachers. For example Hangman, Shut the box, Word chain etc.

We usually start our lecture with experimental work. For example the lesson about congruent triangles starts with a groupwork see figure 3.

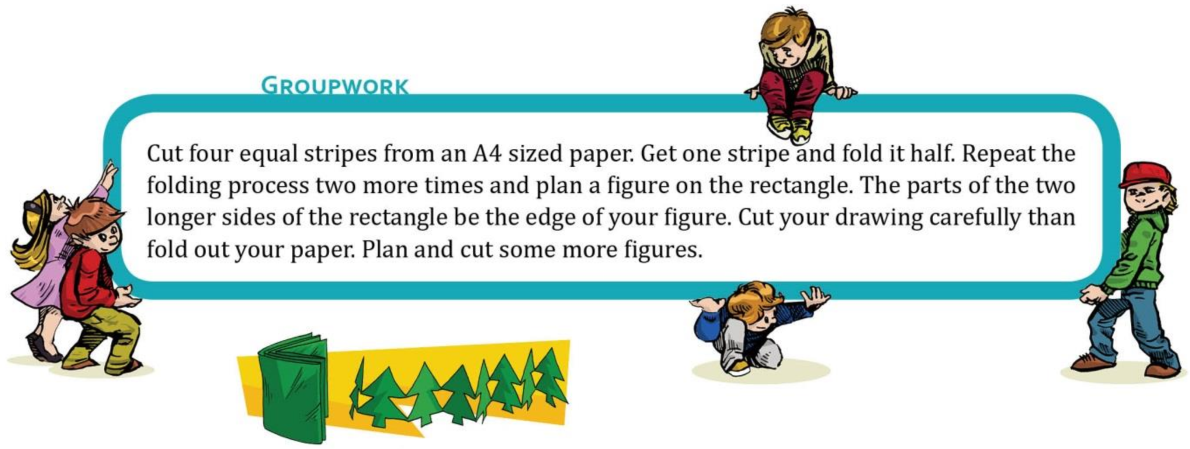

Figure 3*: The group work task from the grade 6. textbook (page 54)*

Media literacy

We applied plenty of different problems to develop the literacy skills of the students. These problems are really easy if somebody is able to process the information from the text and handle it in the approproate way. See the figure 4. This type of literacy problems are really usefull for students and teachers. First they have to read the whole story and only then they can answer the questions. Not only the PISA test but the Hungarian Competency Measure has a lot of similar problems and we have to note that the Hungarian results are unfortunately not on a high level (see Csüllög at all, 2014).

1 The Stonehenge monument is a remaining part of standing stones and earthworks. It was built approximately 2500 BC and finished the work around 2100 BC. Many people think that it is a cultic or astronomical building, erected by the ancient Celts in England. Galileo Galilei found the four big satellites around the Jupiter in 1610 AC and it confirmed his opinion that the Earth is not the centre of the universe.

a) How many years did it take to build Stonhenge?

b) How many years later did Galilei live than the builder of Stonehenge?

c) How many big satellites does the Jupiter have?

d) Look up the planets of the Solar System!

Stonehenge

Figure 4*: Literacy problem from the workbook of grade 5. (page 21)*

Some reactions from teachers feedback:

“There are too few easy tasks to practice, but a lot of word problems.”

“We really need more practicable task for the less talented students.”

We know that it is really useful to have full pages of numbers to sum or multiply them but we also know that practicing without understanding the algorithm behind and the necessity of it is almost useless. We have to find the points which we can remodel and rewrite in order to achieve senseful changes. (When was the last time you had to calculate two ugly fractions without a calculator or a mobile phone?)

Conclusion

We can conclude very useful advices, wishes and comments from the not yet final feedbacks. The teachers on the first hand examined the book from their own viewpoint and only on the second hand hand from the students.

The literacy issues popping up among the teachers confirms that according to the European trends it is crucial to provide high level teacher training and inservice teacher training. The development of high quality textbooks have to go hand in hand with the continuing professional development of teachers.

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I would like to thank for the authors and editors of the mathematics textbooks and workbooks namely Veronika Gedeon, Beáta Tamás, Eszter Paróczay, László Számadó and Anna Szalontay who were not only able to survive my requests, corrections, addition, completion but also supported if we had to cut the material.

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Some Aspects of the Relation between History of Mathematics and Mathematics Education, the Case of Infinite Series

Jozef Fulier*

Department of Mathematics, Faculty of Natural Sciences, Constantine the Philosopher University in Nitra, Tr. A. Hlinku 1, SK-949 74 Nitra,

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Abstract

The paper deals with some aspects of the relationship between history of mathematics and mathematics education. Attention is paid to the importance of integrating elements of history of mathematics in preparation of prospective mathematics teachers. The emphasis is put on personal experience from teaching the university course history of mathematics for prospective teachers and implementation of elements of mathematics history in calculus courses for prospective mathematics teachers. We primarily focus on infinite series because this concept is mysterious and intriguing.

Keywords: History of mathematics, mathematics education, infinite series.

Classification: A30, I30

Introduction

Mathematics in relation to historical periods can be viewed in two ways. On one hand we see it as a system of interconnected timeless eternal facts. In this respect mathematics does not know "outdated" knowledge, in contrast to other natural sciences. Mathematical argument which was once correctly proved (since then it is referred to as a mathematical theorem) never loses this value, although it can happen (and it also occurs in general) that the further development becomes a simple case of more general claim. It cannot, therefore, be surprising that the average person hardly thinks that mathematics has any history, rather, that all of it was revealed in a flash of moment to some *ancient mathematical Muses*. In this sense, it is important to show students that mathematics exists and evolves in time and space. We would like to show that it is a science that has undergone an evolution rather than something which arose out of thin air, and stress that human beings have taken part in this evolution and that the evolution of mathematics has been influenced by many different cultures throughout history and that these cultures have had an influence on the shaping of mathematics as well as the other way round. Mathematicians' activity is focused on the discovery of these timeless facts and clarification of the deductive connection between them. In essence it is about evolutionary progress during which mathematics is becoming better and better. On the other hand, we can look at mathematics as a human activity taking place in the cultural time. This second view is not inconsistent with the first view, quite the opposite,

*Corresponding author: jfulier@ukf.sk
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relevant knowledge effectively complements it by clarifying the logic of discoveries. This allows us to see that the activity with which mathematicians discover new knowledge has much in common with activity that teachers and students do when dealing with the revealed knowledge. Indeed, even in this activity in which teachers are trying to make sense of the learning process, there are elements of creativity and real discoveries. The traditional form of teaching mathematics does not use historical approach in learning. Thus, it is not surprising that at Slovak universities a one-semester course of history of mathematics for prospective mathematics teachers is included in the last year of their master study. The mathematics history lectures in the last year of prospective teacher preparation assumed that history of mathematics cannot enrich the comprehensive knowledge or skill in the subject of mathematics. However, the current mathematics is too abstract and completely separated from its historical roots that may support its understanding. This abstractness of mathematics is also transferred in mathematics education and makes mathematics learning even more difficult. Perhaps a way out of this situation can be to change the view on the relationship of mathematics and its history, and to start systematically investigate the origin and genesis of mathematical ideas, not only in the final course of the history of mathematics, but directly in key university mathematics courses (calculus, algebra, geometry).

The role of mathematics history in mathematics teaching and learning

In general mathematics is viewed as a collection of methods and problems. In our opinion reducing mathematics to this aspect is a distorted picture. Mathematics is much more: it is part of our culture, just as literature, music, philosophy, arts. The cultural aspect of these subjects has been underlined in school by teaching also their historical development. Over the years mathematicians, educators and historians have wondered whether mathematics learning and teaching might profit from integrating elements of history of mathematics. It is clear that mathematics education does not succeed to reach its aims for all students, and that it is therefore worthwhile to investigate whether history can help to improve the situation. The idea to use the history of mathematics in mathematics education is not new. The idea has already been explicitly described in the works of *Heppel*, 1893; *Smith* and *Cajori*, 1894; *Loria* 1899; *Zeuthen,* 1902; *Gebhardt*, 1912. In 1902, Dutch mathematician *H. G. Zeuthen* (1839-1920) published the *History of Mathematics* for teachers. *Zeuthen* argued that the history of mathematics should be part of general teachers' education. Since then, math teachers increasingly use the history of mathematics in their curricula, and the spectrum of its use has spread. Since 1960, research in this area has founded the scientific basis. What is now called "**HPM**" sprang from a Working Group established at the second *International Congress on Mathematical Education* (ICME), held in Exeter, UK, in 1972. The "principal aims" of the Study Group were proposed as follows:

1. To promote international contacts and exchange information concerning: a) Courses in History of Mathematics in Universities, Colleges and Schools. b) The use and relevance of History of Mathematics in mathematics teaching. c) Views on the relation between History of Mathematics and Mathematical Education at all levels. 2. To promote and stimulate interdisciplinary investigation by bringing together all those interested, particularly mathematicians, historians of mathematics, teachers, social scientists and other users of mathematics. 3. To further and deeper understanding of the way mathematics evolves, and the forces which contribute to this evolution. 4. To relate the teaching of mathematics and the history of mathematics teaching to the development of mathematics in ways which assist the improvement of instruction and the development of curricula. 5. To

produce materials which can be used by teachers of mathematics to provide perspectives and to further the critical discussion of the teaching of mathematics. 6. To facilitate access to materials in the history of mathematics and related areas. 7. To promote awareness of the relevance of the history of mathematics for mathematics teaching in mathematicians and teachers. 8. To promote awareness of the history of mathematics as a significant part of the development of cultures.

Educators throughout the world have been formulating and conducting research on the use of history of mathematics in mathematics education. In the last three or four decades there has been a movement towards inclusion of more humanistic elements in the teaching of mathematics. This has been the case of Slovakia, in particular the Slovak upper secondary schools, as well as internationally. The various 'humanistic' elements embrace, among others, cultural, sociological, philosophical, application-oriented, and historical perspectives on mathematics as an educational discipline*. Only recently there has been a stronger call for methodological and theoretical foundations for the role of history in mathematics education. The report *History in Mathematics Education: The ICMI*† *Study* (*Fauvel & Van Maanen*, 2000) has made a valuable contribution in this respect by collecting theories, results, experiences and ideas of implementing history in mathematics education from around the world. Students can experience the subject as a human activity, discovered, invented, changed and extended under the influence of people over time. Instead of seeing mathematics as a ready-made product, they can see that mathematics is a continuously changing and growing body of knowledge to which they can contribute themselves. Learners could acquire a notion of processes and progress and learn about social and cultural influences. Moreover, history accentuates the links between mathematical topics and the role of mathematics in other disciplines, which would help place mathematics in a broader perspective and thus deepen students' understanding. History of mathematics may play an especially important role in the training of future teachers, and also teachers undergoing in-service training. There are several reasons for including a historical component in such training, including the promotion of enthusiasm for mathematics, enabling trainees to see pupils differently, to see mathematics differently, and to develop skills of reading, library use and expository writing which can be neglected in mathematics courses. It may be useful here to distinguish the training needs for primary, secondary and higher levels. (ICMI Study on The role of the history of mathematics in the teaching and learning of mathematics, Discussion Document). History of mathematics provides opportunities for getting a better view of what mathematics is. When a teacher's own perception and understanding of mathematics changes, it affects the way mathematics is taught and consequently the way students perceive it. Teachers may find that information on the development of a mathematical topic makes it easier to explain or give an example to students. *Jankvist* (2009) in his PhD thesis states that in general the arguments for using history are of two different kinds: those that refer to history as a tool for assisting the learning and teaching of mathematics, and those that refer to history as a goal. Each of these two kinds constitutes its own category of arguments. The category of history-as-a-tool arguments

*Note that in the case of Slovakia, this is a little bitter undertone, as efforts to humanize the teaching of mathematics, together with efforts to improve the performance of pupils Slovak testing OECD PISA in 2008 resulted in the reform of educational curriculum, which, though committed to the above mentioned objectives, significantly reduced the number of lessons in individual grades of mathematics education, thus, the initial idea was lost.

†*ICMI = International Commission on Mathematical Instruction*

contains the arguments concerning how students learn mathematics. A typical argument is that history can be a motivating factor for students in their learning and study of mathematics, for instance, helping to sustain the students' interest and excitement in the subject. Or, that an historical approach may give mathematics a more human face and therefore make it less frightening. Often pieces of the mathematical development over which past mathematicians have stumbled are also troublesome for nowadays students of mathematics. Students may derive comfort from knowing that the same mathematical concept which they themselves are now having trouble grasping actually took great mathematicians hundreds of years to shape into its final form. Besides having these motivational and more affective effects, history may also play the role of a cognitive tool in supporting the mathematics learning itself. For instance, one argument states that history can improve learning and teaching by providing a different point of view or mode of presentation. Other arguments say that historical phenomenology may prepare the development of a hypothetical learning trajectory, or that history "can help us look through the eyes of the students". In the context of using history to study learning processes we mention the so-called *Biogenetic Law* popular at the beginning of the 20th century. German biologist and natural philosopher *E. Haeckel* in 1874 formulated his theory as "*Ontogeny recapitulates phylogeny*". The notion later became simply known as *the recapitulation theory*. Ontogeny is the growth (size change) and development (shape change) of an individual organism; phylogeny is the evolutionary history of a species. *Haeckel* claimed that the development of advanced species passes through stages represented by adult organisms of more primitive species. In other words, each successive stage in the development of an individual represents one of the adult forms that appeared in its evolutionary history. Although Haeckel's specific form of recapitulation theory is now discredited among biologists, the strong influence it had on social and educational theories of the late 19th century still resonates in the 21st century. Haeckel developed this thought even further saying that "the psychic development of a child is a brief repetition of the phylogenetic evolution". And it is this argument that translates into the recapitulation argument which may be formulated as: To really learn and master mathematics, one's mind must go through the same stages that mathematics has gone through during its evolution. The Biogenetic Law states that mathematical learning in the individual (*ontogenesis*) follows the same course as the historical development of mathematics itself (*phylogenesis*). Developmental psychologist *Jean Piaget* (1896 – 1980) favoured a softer version of the formula, according to which ontogeny parallels phylogeny because the two are subject to similar external constraints. However, it has become more and more clear since then that such a strong statement cannot be sustained.

The plenary lecture given to the Congress (ICME 4, Berkeley 1980) by the distinguished Dutch mathematics educator *Hans Freudenthal* (1905 – 1990) valuably included his succinct views on the "*ontogeny recapitulates phylogeny*" debate which has long been a concern to those in HPM circles:

Figure 1*:* Jean Piaget (1896 – 1980)

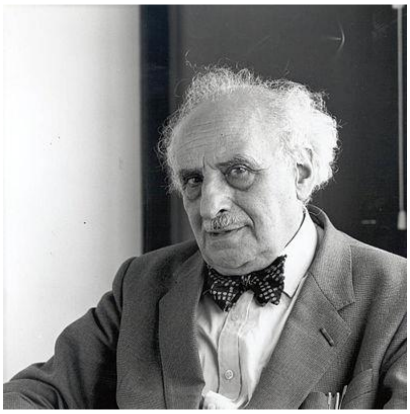

Figure 2*:* Hans Freudenthal (1905 -1990)

"History of mathematics has been a learning process of progressive schematizing. Youngsters need not repeat the history of mankind but they should not be expected either to start at the very point where the preceding generation stopped. In a sense youngsters should repeat history though not the one that actually took place but the one that would have taken place if our ancestors had known what we are fortunate enough to know."(Freudenthal, 1980). A short study of mathematical history is sufficient to conclude that its development is not as consistent as this law would require. *Freudenthal* explains what he understands by "guided reinvention"*: "Urging that ideas are taught genetically does not mean that they should be presented in the order in which they arose, not even with all the deadlocks closed and all the detours cut out. What the blind invented and discovered, the sighted afterwards can tell how it should have been discovered if there had been teachers who had known what we know now. (...) It is not the historical footprints of the inventor we should follow but an improved and better guided course of history."*(*Freudenthal,* 1973).
The recapitulation argument not only applies to mathematics as a whole, but also to single mathematical concepts and theories. And it is often in relation to the development of single mathematical concepts that another tool argument related to the evolutionary kind, the so-called historical parallelism, is put to the "test" – historical parallelism concerns the observation of difficulties and obstacles that occurred in history reappearing in the classroom. The idea of parallelism may also be used as a methodology or heuristic to generate hypotheses in mathematics education (e.g. *Fauvel &van Maanen*, 2000, p. 160;). The German mathematician *Otto Toeplitz*[*] (1881-1940) proposed and distinguished between the "*Direct Genetic Method*" and the "*Indirect Genetic Method*". The "*Indirect Genetic Method*" means that the teacher can learn from the history about difficulties which were encountered even by the great mathematicians such as *Newton, Leibniz, Fermat, Cavalieri* and others and to take this into account in his planning of the teaching process without mentioning historical details.

[*] In 1949 (in German) and 1963 (in English) his textbook *The Calculus A genetic approach* was published posthumously. This book presented a radically different approach to the teaching of calculus. In sharp contrast to the methods of his time, *Otto Toeplitz* did not teach calculus as a static system of techniques and facts to be memorized. Instead, he drew on his knowledge of the history of mathematics and presented calculus as an organic evolution of ideas beginning with the discoveries of Greek scholars, such as *Archimedes, Pythagoras*, and *Euclid*, and developing through the centuries in the work of *Kepler, Galileo, Fermat, Newton,* and *Leibniz*. Through this unique approach, *Toeplitz* summarized and elucidated the major mathematical advances that contributed to modern calculus.

The historical development only acts as a guideline. It shows the teacher (or the textbook author) the crucial way forward: namely, that those aspects of a concept which historically have been recognised and used before others are probably more appropriate for the beginning of teaching than modern deductive reformulations. The genetic method that is going back to the roots of the concepts can offer a way beyond the dilemma of rigour versus intuition in teaching. In contrary to this the "*Direct Genetic Method*" proposes in addition to offer historical details as well (often only a few sentences or a single historical problem) explicitly in the teaching at suitable occasions (*Kronfellner,* 2000). *Schubring* (1978), in his extensive study hereof, distinguishes between two genetic principles: (1) the historical-genetic principle, which aims at leading students from basic to complex knowledge in the same way that mankind has progressed in the history of mathematics, and (2) the psychological-genetic principle, which is based on the idea to let the students rediscover or reinvent mathematics by using their own talent and experiences from the surrounding environment. (*Jankvist*, 2009).

Calculus, infinite series

Calculus fine example of mathematical disciplines in teaching that apply to supplements of *Freudenthal* to Biogenetic Law (mathematical learning in the individual (ontogenesis) follows the same course as the historical development of mathematics itself (phylogenesis) in such a large extent that we can say the teaching of mathematical analysis is currently being implemented under the "*Anti Biogenetic Law*". Indeed, according *Hairer& Wanner* (2008):

"*Traditionally, a rigorous first course in Analysis progresses (more or less) in the following order:* ***sets, mappings ⇒ limits, continuous functions ⇒ derivatives ⇒ integration****. On the other hand, the historical development of these subjects occurred in reverse order:* ***Cantor 1875, Dedekind ⇐ Cauchy 1821, Weierstrass⇐ Newton 1665, Leibniz 1675⇐ Archimedes, Kepler 1615, Fermat 1638."***

University calculus course, especially function and limits of function, cause serious problems to students. On the other hand, it is known from history that terms function and limits of function using *ε-δ notation* were introduced to mathematics in the end of 19^{th} century by German mathematicians *P. G. L. Dirichlet* (1805 -1859) and *K.* Weierstrass (1815 - 1897). The start of using this notation meant the final step in such a very important period that lasted for several centuries. Within this development intuitive and easily understandable terms were substituted by less visual and understandable terms. That is why, as presented by L. Kvasz, lot of students can not translate into their own language everything that they hear in the lectures. The secondary schools mathematics ends up at the level of the 17^{th} century mathematics (with polynomials and systems of equations), but university courses start with the 19^{th} century mathematics notation and language. It means that two hundred years were cut out of syllabus. We also need to mention that this period meant a kind of stagnation for algebra, but for calculus it was a time of rapid development. During this time several approaches to terms like function, limit of function, derivative and integral were used.

For students it means that this approach is hardly understandable because they do not understand the reason why the general term of function and limits of function have been defined by ε-δ notation during the lectures. This gap in understanding can be nicely bridged by history of mathematics, to study the historical development of the terms or theory. Based on this situation we can see the history of mathematics as a bridge that helps prospective teachers to overcame the gap that is between the mathematical concepts developed during secondary education and university courses.

Infinite series

The theory of infinite series is an especially interesting mathematical construct due to its wealth of surprising results. In its most basic setting, infinite series is vehicle we use to extend the finite addition to the "infinite addition". The standard presentation of infinite series in calculus courses in Slovakia as taught today is as follows:

(1) A short introduction to infinite sequences and their limits, convergent and divergent sequences;
(2) Abstract definitions of infinite series $\sum_{n=1}^{\infty} a_n$, with convergence defined in terms of limits of sequences of partial sums $\lim_{n\to\infty} s_n =: s$, where $s_n = a_1 + a_2 + \cdots + a_n, n \in \mathbb{N}$. If $s \in \mathbb{R}$, then we say that series $\sum_{n=1}^{\infty} a_n$ converges to the sum s, and we write $\sum_{n=1}^{\infty} a_n = s$;
(3) Theorems and convergence tests for positive term series; for alternating series and for general series;
(4) Definition and theorems about power series and general functional series.

The emphasis is put on convergence and especially on convergence tests. We spend our time investigating whether series converge or not, but little or no time investigating what the series converge to. The preoccupation with determining convergence but not the sum makes the whole process seem artificial and pointless for many students, and instructors as well. (*Lehmann*, 2000).

At *12th Nitra conference* a lecturer wrote in her lectures on computer graphics some equalities in the form of divergent series (for example $1 + 2 + 3 + \cdots + n + \cdots = -\frac{1}{12}$) derived by *Leonhard Euler* (1707 – 1783). Most of the participating students seemed amused, even shaking their heads in disapproval. The noise which followed and the expression in their faces said: "The equality cannot hold, because these series is divergent. It does not make sense. How is it possible that Euler, one of the greatest mathematicians in history, did not know this? But the first year students already know this!"

This situation can be commented by *D. J. Struik* (1948) that: "we cannot always follow Euler when he writes that $1 - 3 + 5 - 7 + \cdots = 0$, or when he concludes from $n + n^2 + \cdots = \frac{n}{1-n}$, and $1 + \frac{1}{n} + \frac{1}{n^2} + \cdots = \frac{n}{n-1}$ that $\ldots + \frac{1}{n^2} + \frac{1}{n} + 1 + n + n^2 + \cdots = 0$. In this situation we must be careful and not too much criticize Euler for his way of manipulating divergent series; he simply did not always use some of our present tests of convergence or divergence as a criterion for the validity of his series. It is known that despite the incredible amount of work that Euler did, he also wrote occasionally some things that were wrong. Euler's foundation of the calculus may have had some weakness, but he expressed his point of view without vagueness. A lot of his excellent work with series has been given a high credit by modern mathematicians." A partial answer to this question can be found in the history of infinite series. In the first place, it is good to remark that famous definitions of infinite series $\sum_{k=1}^{\infty} a_n$, with convergence defined in terms of limits of sequences of partial sums, was formulated by the excellent French mathematician *Augustin-Louis Cauchy* (1789 - 1857) in his textbook *Cours d'Analyse* in 1821. Almost all of modern definitions of convergence of infinite series copy *Cauchy's* words formulated in his *Cours d'Analyse*:

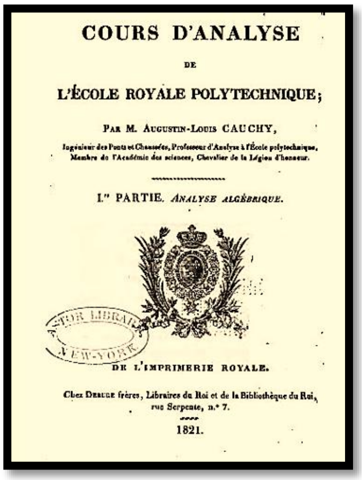

***Figure 3**:* Cours d'Analyse (1821)

***Figure 4**:* Augustin Louis Cauchy (1789 – 1857)

"We call a series an indefinite of quantities, $u_0, u_1, u_2, u_3, \ldots$, which follow from one to another according to a determined law. These quantities themselves are the various terms of the series under consideration. Let $s_n = u_0 + u_1 + u_2 + \cdots + u_{n-1}$ to be the sum of the first n terms, where n denotes any integer number. If, for ever increasing values of n, the sum s_n indefinitely approaches a certain limit s, the series is said to be convergent, and the limit in question is called the sum of the series. On the contrary, if the sum s_n does not approach any fixed limit as it increases infinitely, the series is divergent, and does not have a sum. In either case, the term which corresponds to the index n, that is u_n, is what we call the general term. For the series to be completely determined, it is enough that we give its general term as a function of the index n." (*Bradley & Sandifer*, 2009). Clarity and naturalness of *Cauchy's definition** gives the impression that this is the only way we can define the sum of the infinite series and its convergence. It even seems to be the only available approach. The historical truth is different. First, we must realize that *infinite series* had already more than 2000 years of history at that time, and specific definition of an infinite series, with convergence defined in terms of ***limits of sequences*** of partial sums, was formulated later. Note that *Newton* used the term "*prime and ultimate ratio*" for the "*fluxion*", as the first or last ratio of two quantities just springing into being. *D'Alembert* replaced this notion by the *conception of a limit* in the article "*Limite*" of the "*Encyclopedia*"(edited by *D. Diderot* and until 1759 co-edited by *d'Alembert*) at the end of the 18th century. Moreover, as stated by *N. Bourbaki* (1993) *"And if d'Alembert is happier here, and recognises that in the "metaphysics" of the infinitesimal Calculus there is nothing other than the notion of limit (articles DIFFERENTIEL and LIMITE), he is no more able than his contemporaries, to understand the real meaning of expansion in divergent series, and to explain* ***the paradox of exact results obtained at the end of calculations with expressions deprived of any numerical interpretation**."* We

* Thus we currently define *a series* to be an ordered pair $(\{a_n\}, \{s_n\})$ of sequences connected by the relation $(s_n = \sum_{k=1}^{n} a_k)$ for all $n \in \mathbb{N}$.

can see that in some respect it looks like a task to solve a simple mathematical problem that can be found in *IQ tests*, but also in mathematical textbooks:

Find the next number in the sequence: $2, 4, 6, \ldots.$

It is number 8 which is almost exclusively considered to be the natural and the only correct answer. Students are then very surprised (some authors of IQ tests may not know it now), that task has infinitely many solutions. Any real number $A \in \mathbb{R}$ is the solution of this problem. In other words, each (numerical) answer is correct. Just look at the problem in terms of numerical analysis, construct corresponding Lagrange polynomial a_n and determine a_4:

$a_n = 2\frac{(n-2)(n-3)(n-4)}{(1-2)(1-3)(1-4)} + 4\frac{(n-1)(n-3)(n-4)}{(2-1)(2-3)(2-4)} + 6\frac{(n-1)(n-2)(n-4)}{(3-1)(3-2)(3-4)} + A\frac{(n-1)(n-2)(n-3)}{(4-1)(4-2)(4-3)}, n \in \mathbb{N}.$ We can easily verify that for $n = 4$ we really get $a_4 = A$.

Some historical remarks on infinite series

A time-honoured problem in this area is *Zeno of Elea's* paradoxes of "*Dichotomy*", and "*Achilles and the Tortoise*"*, which are concerned with convergent geometric series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots = 1$. According to *Aristotle,* Zeno's argument is a fallacy. For one cannot actually subdivide an interval infinitely many times. Infinite subdivision is only potential. In his *Physics* Aristotle (384-322 BC) himself implicitly underlined that the sum of a series of infinitely many addends (potentially considered) can be a finite quantity. Of course, it is possible to employ several visual representations (see *Figure 1*: the big square with sides long 1 unit, divided into a sequence of squares or triangles). In his *Quadratura parabolæ* Archimedes (287-212 BC) considered (implicitly, once again) a geometric series $1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots + \frac{1}{4^n} + \cdots = \frac{4}{3}$. Sums of other special geometric series were determined by mathematicians *Nicole Oresme* (1323 - 1382) and *R. Swineshead*. Geometric series played a crucial role in earlier research on series. *Swineshead* in his work (1350) when determining

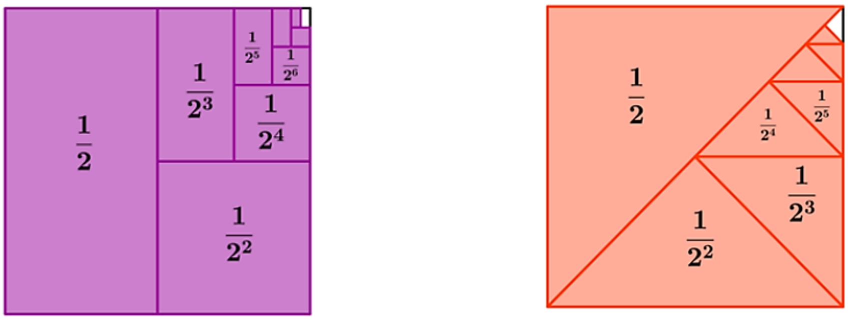

Figure 5*:* Visualizations a sum of series $\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \cdots = 1$.

the average speed of uniformly accelerated motion needed to determine the sum of the first infinite series which was not geometric. He proved that $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n} + \cdots = 2$. *Oresme* was the first who managed to show (around 1350) that the harmonic series

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**Dichotomy Paradox:* That which is in locomotion must arrive at the half-way stage before it arrives at the goal. *Achilles and the Tortoise:* In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead. (*Aristotle, Physics*)

$\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots$ is divergent. It is a very surprising result because the harmonic series diverges very slowly, e. g. the sum of the first 10^{43} terms is less than 100. The main *Oresme's* consideration consisted of an interesting estimate

$1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\cdots+\frac{1}{8}\right)+\left(\frac{1}{9}+\cdots+\frac{1}{16}\right)+\cdots>1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots$. However, one must not conclude that *Oresme* or mathematicians in general began to distinguish convergent and divergent series. His results were lost several centuries, and the result was proved 1647 again by Italian mathematician *Pietro Mengoli* and in 1687 by Swiss mathematician *Johann Bernoulli*. In his *Varia Responsa* (1593) François Viète (1540 -1603) gave the formula for the sum of an infinite geometric progression. From Euclid´s *Elements* he took that the sum of n terms of $s_n=a_1+a_2+\cdots+a_n$ is given by $a_1\colon a_2=(s_n-a_n)\colon(s_n-a_1)$. Then if $\mathrm{a}_1/\mathrm{a}_2>1$, a_n approaches 0 as becomes infinite, so that $s_\infty=\frac{(a_1)^2}{a_1-a_2}$ (*Kline*, 1972). A few decades later *Grégoire de Saint-Vincent* made geometric series a crucial instrument in his method of quadratures. Saint-Vincent, as well Viète, had an intuitive but clear idea of what the sum of series was (whatever words they used to denote the sum). Mengoli made a remarkable contribution to the uprising theory of series. Mengoli found $\sum_{k=1}^{\infty}\frac{(-1)^{n+1}}{n}=\ln 2$ and he showed how it is possible to determine the sum of several infinite series, which we now call the *telescopic series* (specifically $\sum_{k=1}^{\infty}\frac{1}{k(k+m)}$ for $m=1,2,3$). Another important step was made by Isaac Newton (1642 – 1727) in *A Treatise on the Methods of Series and Fluxions* (1671), when infinite series with numbers are extended to series containing variable expressions. Newton asserts that any proper operation that can be performed in arithmetic on numbers can likewise be performed in algebra on variable expressions. Just as arithmetic operations produce highly useful infinite decimal expressions, the same operations may produce highly useful infinite series in algebra. If we compare the following two processes (in modern symbols) that are almost identical, and in both cases the exception of the condition $q\neq 1$, *no restriction on q:*

1. If $s_n=a+aq+aq^2+\cdots+aq^{n-1}$, then $qs_n=aq+aq^2+\cdots+aq^{n-1}+aq^n$, therefore $qs_n-s_n=aq^n-a$, whence $s_n=a\frac{1-q^n}{1-q}, q\neq 1$.
2. If $S=a+aq+aq^2+aq^3+\cdots$, then $qS=aq+aq^2+aq^3+aq^4+\cdots$, therefore $S-qS=a$. Therefore, for $q\neq 1$ it holds $S=\frac{a}{1-q}$.

Using the previous relationships, we also get (for $q=x, q=-x,\ q=-x^2$) the following sums of infinite series: $1+x+x^2+x^3+\cdots=\frac{1}{1-x}, x\neq 1$, respectively $1-x+x^2-x^3+\cdots=\frac{1}{1+x}, x\neq -1, 1-x^2+x^4-x^6+\cdots=\frac{1}{1+x^2}$.

Newton, Euler and Lagrange considered infinite series to be *a part of algebra of polynomials*. It means that series were considered to be polynomials that can express the original function, without any *convergence control*.

Perhaps this is a point where most, if not all, students would agree and they would require no proof that there is anything wrong with this reasoning. From these geometric series a power series representation can be obtained for a wider variety of functions, since power series can be *differentiated* or *integrated term by term* in order to obtain a new power series. Of course in such a formal approach no restrictions on x do not occur. Isaac Newton in 1667 and *N. Mercator* in 1668 obtained the result $\sum_{k=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = \ln(1+x)$ by integrating the power series for $\frac{1}{1+x}$. The sensational discovery of sums of this series just opened quite new perspectives for the application of series, and mainly power series, to problems previously referred to as "impossible". J. Gregory in 1671 and G. W. Leibniz in 1673 obtained the result $\sum_{k=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4}$ by integrating the power series for $\frac{1}{1+x^2}$ and stating $x = 1$.

It is appropriate to note that according to Kline (1972) infinite series were in the 17th and 18th centuries and are still today considered to be the essential part of calculus. Indeed, Newton considered series inseparable from his method of fluxions because the only way he could handle even slightly complicated algebraic functions and the *transcendental functions* was to expand them into infinite series and differentiate or integrate term by term. Newton obtained many series for algebraic and transcendental functions. In his *De Analysi* in 1669 he provided the series for $\sin x, \cos x, \arcsin x, e^x$. The Bernoullis, Euler, and their contemporaries relied heavily on the use of series. Only gradually did the mathematicians learn to work with the *elementary functions in closed form*, that is, simple *analytical expressions*. Nevertheless, series were still the only representation for some functions and the most effective means of calculating the elementary transcendental functions. The successes obtained by using infinite series became numerous as the mathematicians gradually extended their discipline. The difficulties in the new concept were not recognized, at least for a while. Series were just infinite polynomials and appeared to be treatable as such. Moreover, it seemed clear, as Euler and Lagrange believed, that every function could be *expressed in form of a series*.

Intuitive understanding of the concept of the sum of an infinite series and often mechanical transmission of properties of finite sums on infinite sums has brought many problems and paradoxical outcome. Simply said, a finite sum is well-defined, an infinite sum is not. This can be illustrated by a simple example of infinite series.

Let s denote the sum of (telescopic) convergent series $s = \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots$. Then $s = \left(\frac{1}{1} - \frac{2}{3}\right) + \left(\frac{2}{3} - \frac{3}{5}\right) + \left(\frac{3}{5} - \frac{4}{7}\right) + \cdots = \frac{1}{1} - \frac{2}{3} + \frac{2}{3} - \frac{3}{5} + \frac{3}{5} - \frac{4}{7} + \cdots = 1$, since all terms after the first one are mutually cancelled out. Again $\mathrm{s} = \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right) + \cdots = \frac{1}{2} - \frac{1}{6} + \frac{1}{6} - \frac{1}{10} + \frac{1}{10} - \frac{1}{14} + \cdots = \frac{1}{2}$, since all terms after the first are cancelled out. Then $1 = \frac{1}{2}$.

The series which provoked the greatest debates and controversy is $1 - 1 + 1 - 1 + \cdots + (-1)^{n+1} + \cdots$. It seemed clear that by writing the series as $(1-1) + (1-1) + (1-1) + \cdots$ the sum should be 0. It also seemed clear that by writing the series as $1 + (-1+1) + (-1+1) + (-1+1) + \cdots$ the sum should be 1. However, if S is used to denote sum $S = 1 - 1 + 1 - 1 + \cdots$, then $S = 1 - (1 - 1 + 1 - 1 + \cdots)$, i. e. $S = 1 - S$, so $S = \frac{1}{2}$. Guido Grandi (1671 - 1742), a professor of mathematics at University of Pisa, in his book *Quadratura Circuli et Hyperbolae* (1703), obtained the result by another method. He set $x = 1$ in the expansion $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$ and obtained $\frac{1}{2} = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$. *Guido Grandi* considered the formula $\frac{1}{2} = 1 - 1 + 1 - 1 + 1 - 1 + \cdots = (1 - 1) + (1 - 1) + \cdots = 0 + 0 + 0 + \cdots = 0$, to be the symbol for creation of world from *Nothing*. He obtained the result $\frac{1}{2}$ by considering the case of a father who bequeathed a gem to his two sons, each allowed to keep it one year alternately. It then belonged to each son by one half. (*Struik*, 1948). Several mathematicians of this period (*J. Riccati, P. Varignon* and *Nicholas I. Bernoulli*) did not agree with this reasoning. The same result, based on interesting probabilistic reasoning, was arrived at by G. W. Leibniz. Instead, Leibniz argued that if one takes the first term, the sum of the first two, the sum of the first three, and so forth, one obtains $1, 0, 1, 0, 1, \ldots$. Thus 1 and 0 are equally probable; one should therefore take the arithmetic mean, which is also the most probable value, as the sum. This solution was accepted by James and John Bernoulli*,* Daniel Bernoulli, and Lagrange*.* Leibniz conceded that "his argument was more metaphysical than mathematical, but went on to say that there was more metaphysical truth in mathematics than was generally recognized" (*Kline*, 1972, p. 446). *Christian Wolf* (1678 -1754) wished to conclude that $1 - 2 + 4 - 8 + \cdots = \frac{1}{3}$, $1 - 3 + 9 - 27 + \cdots = 1/4$ by using an extension of Leibniz's own probability argument. Real extensive work on series began about 1730 with *Leonhard Euler* (1707-1783), who aroused tremendous interest in the subject. *Euler* summarizes Leibniz's arguments and he wrote: "Now if, therefore, the series is taken to infinity and (consequently) the number of terms cannot be regarded as either even or odd, it cannot be concluded that the sum is either 0 or 1, but we ought to take a certain median value which differs equally from both, namely ½." To obtain the sum of $1 - 1 + 1 - 1 + \cdots$. His second argument pointed out in his textbook *Institutiones calculi differentialis* (1755):

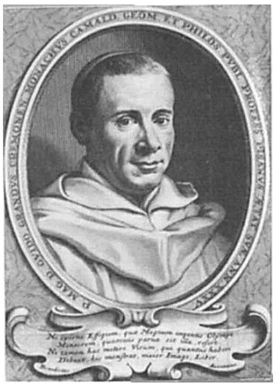

Figure 6*:* Luigi Quido Grandi (1671 -1742)

Figure 7*:* Leonhard Euler (1707 - 1783)

"We state that the sum of an infinite series is the finite expression by which the series is generated. From this point of view the sum of the infinite series $1 - x + x^2 - x^3 + \cdots$ is $1/(1 + x)$ because the series arises from the development of the fraction, for every value x."

As divergent series are considered such an inconvenience, we can settle defining the sum of a series in terms of limits of sequences of partial sums (as we do at present) and dismiss as "divergent" any series that does not satisfy this convergence requirement. ***Well, the lecturer, who presented at the Conference in Nitra, knew well Cauchy´s definition of sum of infinite series.*** Another alternative is to redefine the concept of "sum". Among those making the attempt to save the *divergent series* for analysis was Leonhard Euler. He attempted to redefine the meaning of "sum" in a significantly more abstract fashion, further from the then common understanding of "*sum*" as "*to add up*." Euler proposed the following definition for "sum", also referred *Euler's principle*: *"Understanding of the question is to be sought in the word "sum"; this idea, if thus conceived-namely, the sum of a series is said to be that quantity to which it is brought closer as more terms of the series are taken-has relevance only for convergent series, and we should in general give up this idea of sum for divergent series. Wherefore, those who thus define a sum cannot be blamed if they claim they are unable to assign a sum to a series. On the other hand, as series in analysis arise from the expansion of fractions or irrational quantities or even of transcendentals, it will in turn be permissible in calculations to substitute in place of such a series that quantity out of whose development it is produced. For this reason, if we employ this definition of sum, that is, to say the sum of a series is that quantity which generates the series, all doubts with respect to divergent series vanish and no further controversy remains on this score, inasmuch as this definition is applicable equally to convergent or divergent series."* (*Barbeau and Leah*, 1976).

The intuitive formation of this definition of "sum" reflects an attitude still current among applied mathematicians and physicists: problems that arise naturally (i.e., from nature) do have solutions, so the assumption that things will work out eventually is justified experimentally without the need for existence sorts of proof. Assume everything is okay, and if the arrived-at solution works, you were probably right, or at least right enough. (Lehmann, 2000). For example *Euler* wrote

$$1/4 = 1 - 2 + 3 - 4 + \cdots, \qquad 0 = 1 - 3 + 5 - 7 + \cdots, \qquad -1 = 1 + 2 + 4 + 8 + \cdots$$

because these series arose from expansions

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \cdots, \frac{1-x}{(1+x)^2} = 1 - 3x + 5 - 7x^3 + \cdots, \frac{1}{1-2x} = 1 + 2x + 4x^2 + \cdots$$

and setting $x = 1$.

N. Bernoulli replied that the same series might arise from expansion of two different functions and, if so, the sum would not be unique. From a theoretical point of view this would be a serious problem. And, really *J. Ch. Callet* showed that the series $1 - 1 + 1 - 1 + \cdots$ may be obtained from the expansion $\frac{1+x}{1+x+x^2} = \frac{1-x^2}{1-x^3} = 1 - x^2 + x^3 - x^5 + x^6 - x^8 + \cdots$ and setting $x = 1$, we get $\frac{2}{3}$ instead Euler's $\frac{1}{2}$. *Joseph-Louis L. Lagrange* (1736 -1813) considered this objection and argued that Callet's example was incomplete. When the missing terms were included, the series should have been written

$$1 + 0x - x^2 + x^3 + 0x^4 - x^5 + x^6 + 0x^7 - x^8 + \cdots$$

so what was summed was $1 + 0 - 1 + 1 + 0 - 1 + 1 + 0 - 1 + \cdots$ a series whose partial sums are $1, 1, 0, 1, 1, 0, \ldots$ with average sums $\frac{2}{3}$. And, in fact, Euler's assertion, when properly interpreted, is correct, since a convergent power series has a unique generating function. It is clear that before the 19th century divergent series were widely used by Euler and others, but often led to confusing and contradictory results. English mathematician *G. H. Hardy* (1877-1947), author of the excellent book *Divergent series* (1949) suggests that "*...It is a mistake to think of Euler as a 'loose' mathematician, though his language may sometimes seem loose to modern ears; and even his language sometimes suggests a point of view far in advance of the general ideas of his time. …Here, as elsewhere, Euler was substantially right. The puzzles of the time about divergent series arose mostly, not from any particular mystery in divergent series as such, but from disinclination to give formal definitions and from the inadequacy of the current theory of functions. It is impossible to state Euler's principle accurately without clear ideas about functions of a complex variable and analytic continuation*". It must, however, admit that in spite of great Euler's authority divergent series arouse ever greater mistrust. This attitude of mathematicians is succinctly explained by Norwegian mathematician *Niels H. Abel* (1802 – 1829) in 1828: "*Divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever*." In the ensuing period of critical revision they were simply rejected. Then came a time when it was found that something after all could be done about them. Mathematics after Euler moved slowly but steadily towards the orthodoxy ultimately imposed on it by *Cauchy*, *Abel*, and their successors, and divergent series were gradually banished from analysis, to appear only in quite modern times. After *Cauchy*, the opposition seemed definitely to have won.

The above mentioned information should not lead to the impression that Cauchy's *Cours d'Analyse* textbook, which already contains the foundations of his new theory of infinite series, was a step backwards. The opposite is true. It was a very necessary and important step. *Cauchy* was in fact another brilliant mathematician who greatly influenced the character of infinitesimal calculus. Victory of the Cauchy's approach was very important for the further development of the actual theory of infinite series. Cauchy's formulation of the definition of the sum of the infinite series by a sequence of partial sums limits and precise distinction between convergent and divergent series, was a ground-breaking milestone in the history of the theory of series. This has become the standard for the next period, providing a uniform platform and some unified approach for justifying and derivation of results.

Cauchy's attitude to divergent series is openly declared in the introduction to the Cours d'Analyse: "*As for methods, I have sought to give them all the rigor that one requires in geometry, so as never to have recourse to the reasons drawn from the generality of algebra. Reasons of this kind, although commonly admitted, particularly in the passage from convergent series to divergent series, and from real quantities to imaginary expressions, can, it seems to me, only sometimes be considered as inductions suitable for presenting the truth, but which are little suited to the exactitude so vaunted in the mathematical sciences. We must at the same time observe that they tend to attribute an indefinite extension to algebraic formulas, whereas in reality the larger part of these formulas exist only under certain conditions and for certain values of the quantities that they contain. Determining these conditions and these values, and fixing in a precise way the sense of the notations I use, I make any uncertainty vanish; and then the different formulas involve nothing more than relations* *among real quantities, relations which are always easy to verify on substituting numbers for the quantities themselves. In order to remain faithful to these principles, I admit that I was forced to accept several propositions which seem slightly hard at first sight. For example . . . a divergent series has no sum*." (*Cauchy*, 1821, ii–iii). The initial reaction of our founders of nineteenth-century analysis (Cauchy, Abel, and others) was that valid arguments could be based only on convergent series. Divergent series were mostly excluded from mathematics. We saw that many eighteenth century mathematicians achieved spectacular results with divergent series but without a proper understanding of what they were doing. They reappeared in 1886 with *Poincare's* work on asymptotic series. In 1890 *Ernesto Cesàro* (1859-1906) realized that one could give a rigorous definition of the sum of some divergent series, and defined *Cesàro summation* as follows:

If $s_n = a_1 + a_2 + \cdots + a_n$ and $\lim_{n\to\infty} \frac{s_1+s_2+\cdots+s_n}{n} := s \in \mathbb{R}$, then we call s the $(C,1)$ sum of $\sum_{n=1}^{\infty} a_n$ and we write $\sum_{n=1}^{\infty} a_n = s[\text{Cesàro}]$.

For example $\sum_{n=1}^{\infty}(-1)^{n+1} = \frac{1}{2}[\text{Cesàro}]$, but $\sum_{n=1}^{\infty} n$ is not Cesàro summable because the terms of the sequence of means of partial sums $\{t_n\}$, $t_n = \frac{1}{n}\sum_{k=1}^{n} s_k$ are here $\frac{1}{1}, \frac{4}{2}, \frac{10}{3}, \frac{20}{4}, \ldots$ and this sequence diverges to infinity. Cesàro's key contribution was not the discovery of this method but his idea that one should give an explicit definition of the sum of a divergent series. In the years after Cesàro's paper several other mathematicians gave other definitions of the sum of a divergent series, though these are not always compatible: different definitions can give different answers for the sum of the same divergent series, so when talking about the sum of a divergent series it is necessary to specify which summation method one is using. In addition to Cesàro summation to the most known summation methods include Abel summation and Euler summation. For illustration we add Abel summation method which is similar to the well-known *Abel's theorem* on power series:

If $\sum_{n=1}^{\infty} a_n x^n$ is convergent for $0 \le x < 1$ (and so for all x, with $|x| < 1$), $f(x)$ is its sum, and $\lim_{x\to 1-0} f(x) = s$, then we call s the $A\,sum$ of $\sum_{n=1}^{\infty} a_n x^n$ and we write $\sum_{n=1}^{\infty} a_n = s[\text{Abel}]$.

Abel summation is interesting in part because it is consistent with and at the same time more powerful than *Cesàro summation.* We say that *asummability method* M is *regular* if it is equal to the commonly used limit (of partial sums) on all convergent series. It can be easily verified that the *Cesàro summation* and *Abel summation are regular methods.* Such a result is called an abelian theorem for M, from the prototypical Abel's theorem. More interesting and in general more subtle are partial converse results, called *tauberian theorems*, from a prototype proved by Austrian mathematician *Alfred Tauber**. Here partial converse means that if M sums the series Σ, and some side-condition holds, then Σ was convergent in the first place; without any side condition such a result would say that M only summed convergent series (making it useless as a summation method for divergent series). Finally, let us note,

* Tauber was born in Pressburg, now *Bratislava, Slovakia* in 1866 and died in concentration camp in Theresienstadt, now *Terezin, Czech Republic* in around 1942.

using *Hanh-Banach theorem* which is a central tool in functional analysis, ***it can be shown that each series whose sequence of partial sums is bounded is summable through some methods.*** Unfortunately proof of this statement is non-constructive. Mathematicians introduced recurrent series and emphasized the law of formation of coefficients, independent of the convergence of series. The attempt to increase the speed of convergence of series subsequently led to the emergence of asymptotic series, which showed the possibility of using divergent series to obtain appropriate approximations.

In conclusion, let us get back to the infinite series $\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + \cdots$ which we started our considerations with. This series belongs to infinite series which are assumed sums only with tough efforts, since this series is neither *Cesàro summable*, nor *Abel summable.* Using the equation $\sum_{n=1}^{\infty}(-1)^{n+1} n x^n = \frac{1}{(1+x)^2}$, it can be rigorously proved that

$$\sum_{n=1}^{\infty}(-1)^{n+1} n = \lim_{x\to 1^-} \sum_{n=1}^{\infty}(-1)^{n+1} n x^n = \lim_{x\to 1^-} \frac{1}{(1+x)^2} = \frac{1}{4}\,[\text{Abel}].$$

Euler then went on to compute the sum of all natural numbers, as follows. First, he considered what is now called the *Riemann zeta function*[*]:

$$\zeta(s) = 1^{-s} + 2^{-s} + 3^{-s} + 4^{-s} + 5^{-s} + 6^{-s} + \cdots.$$

Multiplying by 2^{1-s}, he obtained $2^{1-s}\zeta(s) = \quad 2.2^{-s} + \quad 2.4^{-s} + \quad 2.6^{-s} + \cdots$. Subtracting the second equation form the first one, he got

$$(1 - 2^{1-s})\zeta(s) = 1^{-s} - 2^{-s} + 3^{-s} - 4^{-s} + 5^{-s} - 6^{-s} + \cdots,$$

then, after evaluating both sides at $s = -1$, he got

$$\zeta(-1) = 1 + 2 + 3 + 4 + \cdots = \left(-\frac{1}{3}\right)(1^{-s} - 2^{-s} + 3^{-s} - 4^{-s} + 5^{-s} - 6^{-s} + \cdots) = -\frac{1}{12}.$$

Conclusions

History of mathematics can deepen the conceptual understanding of mathematical concepts and theories and lighten their roots and sources. This understanding is essential in education as well as in further practice and support for good decision-making. In our contribution we present one basic example of the use of history of mathematics to help lecturers as well as learners understand and overcome epistemological obstacles in the development of mathematical understanding of series. Based on the principal that "*ontogeny recapitulates phylogeny*" – we see it appropriate that the development of an individual's mathematical understanding respect the historical development of mathematical ideas. Even though this approach is demanding in many ways we see this combination with historical and psychological perspectives very promising in further teacher development.

We presented this approach with a short overview of history of series theory. We would like to stress that divergent series should be treated with the same respect as convergent series. The first course in series methods often gives the impression of obsession with the issue of convergence or divergence of a series. The huge amount of tests might lead one to this conclusion. Accordingly, you may have decided that convergent series are useful and proper tools of analysis while divergent series are useless and without merit. In fact divergent series are, in many instances, as important as, or more important than convergent ones. Many eighteenth-century mathematicians achieved spectacular results with divergent series but

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[**] The *Riemann zeta function* is an extremely important special function of mathematics and physics that arises in definite integration and is intimately related with very deep results surrounding the prime number theorem. While many of the properties of this function have been investigated, there remain important fundamental conjectures (most notably the *Riemann hypothesis*) that remain unproved to this day.

without proper understanding of what they were doing. The initial reaction of the founders of the nineteenth-century analysis (Cauchy, Abel, and others) was that valid arguments could be based only on convergent series, and that divergent series should be avoided. There are many useful ways of doing rigorous work with divergent series. One way, which we now study, is the development of summability methods.

This approach also needs to be more emphasized during prospective teacher preparation and teachers should transfer the approach into their teaching.

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